# *Honors* Analysis

# *Chapter 4* Applications

# *2003*

**Exam Chapter 4 Mech**

# 1 R/C Pg 158 - 163

# 2 Explain in **great detail** the origin of **each** of the equations listed in the oral exercises on Pg 163

4 / 25

# 3 Pg 163 1 - 6

# 4 Pg 164 7 - 15

**Lecture: "Something has to be x"**

4/ 28

# 5 Pg 164 16 - 21

Do NOT use method of examples.

Use chain rule

# 6 Pg 184 17, 18

**example 3 pg 181 -182**

4 / 29

# 7 Pg 184 19 , 20

# 8 Pg 188 7, 9, 10

**do pblm 19 using x as variable instead of r**

**Important Lecture**

4 / 30

# 9 Pg 189 11, 12

# 10 "Applying the Derivative"

part I ONLY

**Don't collect yesterday's HW yet.**

5 / 1

# 11 "A the D" 2 – 4

5 / 2

# 12 "The Hamburger Stand"

**Important Lecture**

5 / 5

# 13 "A the D" 5 - 7

Use your grapher to solve the cubic equation in problem 7

5 / 6

# 14 "A the D" 8 - 10

5/7

# 15 "A the D" 11 - 13

5/8

# 16 "A the D" 14 - 16

5/9

# 19 "A the D" 17 - 19

5/12

# 20 "A the D" 20 - 22

5/ 13

# 21 "A the D" 23 - 24

5/14

***You must include a brief explanation of your method for each problem***

Put Last Year's Exam on the Chalkboards

5/15

**Exam Ch 4 *Applications* 2003**

5/16

The scenario: Jake owns a hamburger stand. He had a study done of the people in this area and their desires. The study predicts that He will sell x hamburgers if he charges Q dollars for each hamburger.

The ***Demand Function*** is: $Q = \frac{60000 - x}{20000} = 3 - \frac{x}{20000}$

Jake pays $5,000 a month for his rent and personnel. Each hamburger he sell costs him 56 cents in materials.

the Cost function ( $C$ )is : $C(x) = 5000 + .56x$

the Revenue function ( $xQ$ ) is: $R(x) = 3x - \frac{x^2}{20000}$

the Profit function ( $R - C$ ) is: $P(x) = 3x - \frac{x^2}{20000} - (5000 + .56x) =$

the Marginal Cost function ( $C'$ ) is: $.56$

This is reasonable since he uses 56 cents worth of materials to make ONE hamburger.

the Marginal Revenue function ( $R'$ ) is: $R'(x) = 3 - \frac{x}{10000}$

the Marginal Profit function ( $P'$ ) is: $P'(x) = 2.44 - \frac{x}{10000}$

Let us consider a few possible situations.:

**TABLE #1**

| Number of burgers –> | 10K | 20K | 24,400 | 30K | 40K | 50K |
|---|---|---|---|---|---|---|
| Q = Price/burger | 2.50 | 2.00 | 1.78 | 1.50 | 1.00 | 0.50 |
| R = Total Revenue | 25K | 40K | 43,432 | 45K | 40K | 25K |
| C = Total Costs | 10.6K | 16.2K | 18,664 | 21.8K | 27.4K | 33K |
| P = Total Profit | 14.4K | 23.8K | 24,768 | 23.2K | 12.6K | –8K |

1) Notice that from 10K to 20K, he doubled the number of burgers sold but that did not double his profits.
2) Notice that producing 30K burgers will produce less profit than producing 20K burgers. Why does this occur?
3) We expect there to be a point when Jake would achieve maximum profit.

$$P(x) = 2.44x - \frac{x^2}{20000} - 5000$$

$$P'(x) = 2.44 - \frac{x}{10000} \overset{set}{=} 0$$

$$x = 24,400$$

$$P''(x) = -\frac{1}{10000} < 0 \text{ for the critical value so I have a max. } f'' \text{ test.}$$

Fill in **Table # 1** with the following data using x = 24,400 in the above equations to get: Price per burger is $ 1.78 ; total revenue is $ 43,432 ; total costs are $ 18,664 ; and total profit is $ 24,768.

This is filled in during explanation of the problem.

**TABLE # 2**

| Number of burgers –> | 10K | 20K | 24,400 | 30K | 40K | 50K |
|---|---|---|---|---|---|---|
| MR = R ' | 2 | 1 | .56 | 0 | –1 | –2 |
| MC = C ' | .56 | .56 | .56 | .56 | .56 | .56 |
| MP = P ' | 1.44 | .44 | 0 | –.56 | –1.56 | –2.56 |

Let us look a a slightly different situation. For ease in computation, we shall use the situation when we wish to sell our hamburgers at $ 2.00 each:

From table # 1

We expect to sell 20,000 hamburgers.
We expect to have a gross revenue of $ 40,000.
We expect our expenses to be $ 16,200.
We expect our profit to be $ 23,800.

Look at table # 2.

QUESTION: Do we really expect our marginal revenue to be $ 1.00 when $x = 20{,}000$ ?

Remember, from our definitions page, Marginal Revenue is the "approximate revenue expected for selling ONE more unit."

Restated: Billy Joe-Bob, Jake's trusted employee, hands out burger number 20,000 of the month.
He collects $ 2.00.
Should Jake expect his revenue to increase by $ 2.00 (the money Billy Joe-Bob collected for the burger) or by $ 1.00 (which the Marginal Revenue function says he should expect)?

The marginal costs is 56 cents. Exactly the cost for making that ONE more burger.

The Marginal Profit is 44 cents, the difference between the $ 1.00 Marginal Revenue and the 56 cents of Marginal Costs.

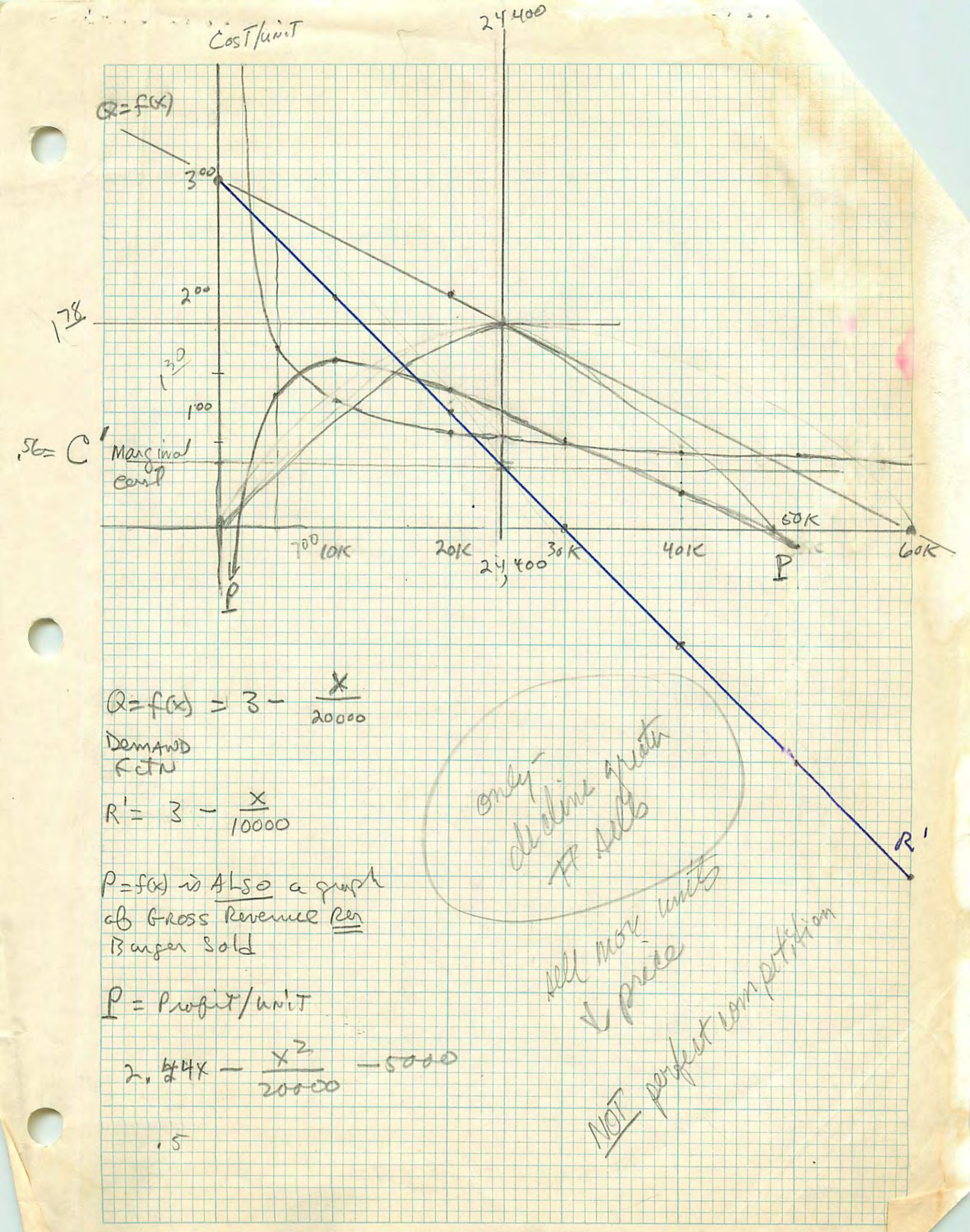
HOMEWORK:

1) Should Billy Joe-Bob just put the "extra" dollar in his pocket?

2) Does the adage "Approximate amount of revenue one should expect to receive with the sales of ONE more unit." fail?

3) Mathematically, everything looks proper. Is the problem here in the concept of the selling price?

4) Resolve the dilemma.




$Q = f(x) = 3 - \frac{x}{20000}$

DEMAND FCTN

$R' = 3 - \frac{x}{10000}$

$P = f(x)$ ⇒ ALSO a graph of GROSS Revenue Per Burger Sold

$\underline{P}$ = Profit/unit

$2.44x - \frac{x^2}{20000} - 5000$

.5

only dealing with greater than 7K sells

sell more units ↓ price

NOT perfect competition

ur investigation might have us fill in Table #2 with the figures for 24,400 burgers sold.
$R' = .56$, $C' = .56$, and $P' = 0$.

Consider the number of burgers to be 19,999.

| | 19,999 | 20,000 |
|---|---|---|
| Q | 2.00005 | 2.00 |
| R | 39.999 | 40,000 |
| C | 16199.44 | 16200.00 |
| P | 23799.56 | 23800.00 |
| R′ | 1.0001 | 1.00 |
| C′ | 0.56 | 0.56 |
| P′ | 0.4401 | 0.44 |

My revenue DID increase only one dollar! My "approximate revenue for one more unit" is correct !

Several economics textbooks point out that marginal revenue should be equal to price per unit. However, each of their examples has a demand which is constant !

The answer lies in the fact that average revenue DOES NOT necessairly equal marginal revenue AND that we do NOT have a varying demand !

I must reduce my price per burger to sell more burgers. By assuming that my demand function is NOT constant I am assuming that burger number 20,000 sold for $2.00 AND that burger number 19,999 sold for $ 2.00005 and burger number 19,998 sold for $ 2.00010 etc.

hen I decide to sell 20,000 burgers, the demand function says I can sell them at $ 2.00 each. That is, once I have decided upon a sales price, my demand function can be disregarded and NOW I have a horizontal, ie. constant, demand function !

Every burger, ie #1, #123, #1,234, #12,345 and #20,000 will sell at $ 2.00 ! My average revenue per burger does equal my selling price !

The actual use of the demand function is three-fold:

1) It allows speculation about the market size. Should I have an operation capable of 10,000 units, 20,000 units or more?

2) It tells me what price I should use to stimulate a certain demand.

3) It allows me to write my revenue function.

Once these three things have taken place, the demand function ceases to have significance.

We expect our marginal revenue to slow when we are very close to our maximum profit. Remember, Marginal Revenue is really nothing more than the slope of the revenue curve. In our example, that revenue curve is a concave down parabola. It has a maximum. When we near that maximum, the vertical change should be very slight.

Finally, when Joe sold burger number 20,000, the cash register had $ 39,998. I expect it to have $ 40,000 fter the 20,000th burger has been sold. Joe should NOT put that "extra" dollar in his pocket.

**I** Given:

| | $f$ | $f'$ | $g$ | $g'$ | $h$ | $h'$ |
|---|---|---|---|---|---|---|
| 0 | 1 | 2 | 2 | 2 | 1 | 2 |
| 1 | 2 | 1 | 1 | 0 | 2 | 1 |
| 2 | 2 | 1 | 2 | 1 | 0 | 0 |

evaluate the following:

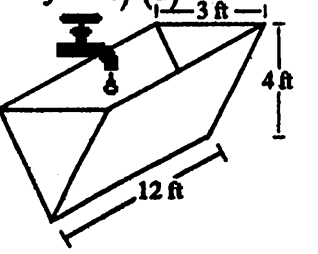
Of course, we know ... $(f \circ g)(x) = f(g(x))$

a) $(f \circ g)'(0)$  b) $(f \circ g)'(2)$  c) $(f \circ h)'(0)$  d) $(g \circ f)'(1)$

e) $(f \circ g)'(1)$  f) $(g \circ g)'(1)$  g) $(g \circ f)'(2)$  h) $(g \circ f \circ h)'(2)$

i) $(f \circ h \circ g)'(1)$  j) $(g \circ h \circ f)'(0)$  k) $(f \circ f \circ f)'(0)$

**II** The length of the water trough is 12 feet. It was made by bending some metal in the shape of a V and then the ends were soldered to the trough to make it water tight. The cross section of the trough ( i.e. each end of the trough) is an isosceles triangle with measurements as shown in the picture. The water faucet delivers water at 3 cubic feet per minute.




A) Assume that the trough has already been filled to a depth of 2 feet ( as measured from the apex of the triangle to the surface of the water.) Your watch reads 2:20 in the afternoon. What time will it be when the trough is completely full?

B) How fast was the depth changing at 2:38?

C) How fast was the depth changing at 2:40?

**III** A package to be sent in the mail can have a combined girth ( distance around) plus length of at most 72 inches. What is the volume of the largest box ( maximum volume ), with a square base, which can be sent?

**IV** Find the extrema of $f(x) = 4x^3 - 12x + 7$ for $[-2, 0]$

**V** Along the graph of the equation $y = x^3 - 6x^2 + 3x + 5$ both the ordinate "y" and the slope ( of the tangent) "$m$" change, but generally at different rates. Find the point (or points), if any, where the ordinate and the slope are momentarily changing at the *SAME* rate.

**VI** Water flows at the rate of 2 cubic feet per minute into a vessel in the form of an inverted right circular cone of altitude 2 feet and base radius 1 foot. At what rate is the surface of the water rising when the vessel is one-eighth full?

**VII** Since 1980 the enrollment at Vacaville High School is approximated by the formula given as "V". The student population of Wood High School is given by the formula "W". (a) What will be the maximum (or minimum) enrollments for each school? (b) When will it occur? ( t is the number of years since 1980 so 1981 is $t = 1$) (c) What are the current enrollments? (d) When, if ever, did (or will) the student populations be the same? (e) Give a reason why the growth rates are reasonable. (f) During which year will both schools have the same number of students? (Please be accurate to the nearest year.)

$$V(t) = 1100 + (t - 17)^2 \qquad W(t) = 1400 - 1000(t + 7)^{-1}$$

**VIII** One leg (AC) of right triangle ACB increases at the rate of 2 in/sec while the other leg decreases at the rate of 3 in/sec. Determine how fast the hypotenuse is changing AND state whether it is getting longer or shorter when leg BC = 6 and leg AC = 8. What about the area of the triangle at the same time?

**IX** Find the altitude of the isosceles triangle with largest area which can be inscribed inside a circle with radius 5.

**X** A man is flying a kite which is a constant 50 feet above the ground. A horizontal wind is blowing the kite away from the man. When 130 feet of string has been let out, the kite is pulling the string out at a rate of 20 ft/sec. What is the wind velocity?

**X** We are considering investing a large sum of money. Our broker brought us the figures for two companies. Acme Unlimited and Baker Industries. The net assets are in units of 1 billion dollars.

$$\text{Acme } A(t) = 3t(t+1)^{-1} \qquad \text{Baker } B(t) = t^2(t+1)^{-1} + 1$$

A) Find the assets for each company one year from today ( ie. when $t = 1$ ).

B) Which company should we invest with? Why?

C) Determine the growth rate ( ie. the rate of change in the assets = growth rate) for each company.

D) Evaluate each of these at $t = 1$.

E) Are you now able to decide which company is the best choice (from the investor's point of view)? Why?

F) Let's consider how the "growth rate" is changing. Since the first derivative of assets is growth rate, it seems logical that the first derivative of growth rate will tell us how active the growth rate is. So find the second derivative of each function.

G) Evaluate both at $t = 1$.

H) When the second derivative is positive, we say the business is growing ( ie. the growth rate is accelerating). ( We are holding the gas pedal of the car completely down.) When the second derivative is negative we say the growth rate is decelerating. ( To continue the analogy, this is like removing your foot from the gas pedal. The car is still traveling but is slowing at the same time.) We have seen that the assets of both companies will be the same 1 year from now, and their growth rates are the same. However, we can see that one of the businesses will increase as time marches on while the other ...

I) Should we choose a company with a currently high growth rate that is decreasing slowly, or one with a smaller growth rate that is increasing slowly? Why?

J) Speculate on the idea that the business which is decreasing has currently a higher return than the business which is increasing -- that is -- which would be the best choice? Why?

**XII** The $(x, y)$ coordinate of a point $P$ is given by: $x = t + \frac{1}{t}$; $y = t - \frac{1}{t}$

A) Find $\frac{dy}{dx}$ (Hint: you are using the definition of *Chain Rule*)

B) Find $\frac{d^2y}{dx^2}$ (This is the second derivative of $y$ with respect to $x$. Answer is **NOT** $-1$)

C) Combine the two given equations in such a fashion that the variable $t$ is eliminated.

D) What conic section is this? Draw its graph -- ( Ask someone in Alg II -- They can help. )

**XIII** The sum of two nonnegative numbers is 1.

A) The sum of the square of the first and the cube of the second is a maximum. Find both numbers.
B) The sum of the square of the first and the cube of the second is a minimum. Find both numbers.

**XIV** Given rectangle $ABCD$ has 10 sq ft of area. Point $E$ is the midpoint of side $BC$. Determine the dimensions of $ABCD$ such that the distance from $A$ to $E$ is a minimum.

**XV** Given the parabola: $y = 16 - x^2$. A rectangle has two of its vertices on the $x$-axis and the other two above the $x$-axis and on the parabola.

A) Of all such possible rectangles, find the dimensions of the one with maximum area.
B) Of all such possible rectangles, find the dimensions of the one with maximum perimeter.

**XVI** To study the effect of production levels on costs, economists use the "average cost function". The average cost function (called "C-bar") is written $\overline{C} = \frac{C(x)}{x}$ where $C(x)$ is the total cost function.

A company estimates that the cost (in dollars) of producing x units of a certain product is given by:

A) $C(x) = 800 + .4x + .0002x^2$

1. Find the production level that minimizes the average cost per unit.
2. Compare the answer to part 1 with the average cost when
   a) 4000 units are produced.
   b) 1000 units are produced.

B) $C(x) = x^3 - 6x^2 + 13x$

1) Find the value of $x$ where the average cost is a minimum.
2) For the value found in part 1, show that the marginal cost and average cost are equal.

C) $C(x) = 2x^2 + 5x + 18$

1) Find the value of x where the average cost is a minimum.
2) For the value found in part 1, show that the marginal cost and average cost are equal.

D) Show that the average costs is a minimum at the value of x where the average costs equals the marginal cost.

**XVII** A business, in marketing a certain item, has discovered through trial and error experiments in several test areas that the demand this item has is given by the equation: $x = \frac{2500}{q^2}$ where $x$ is the number of units produced and $q$ is the price per unit. Assuming that the total revenue function is normal (i.e. $R = x\,q$) and the cost function is $C(x) = .5x + 500$, find the price per unit that yields a maximum profit.

**XVIII** The manufacturer of Sweet Bippies charges \$90 per sweet bippy while the cost per sweet bippy is \$60. To encourage larger orders from the distributors, the manufacturer will reduce the price per Sweet Bippy by 10 cents for each sweet bippy ordered in excess of 100. (For example, there would be a charge of \$88 per bippy for an order size of 120). What would be the ideal order size, from the manufacturer's point of view. What would happen if an order came in which was twice as big as that?

**XIX** The vertices of a triangle are $(0, 0)$, $(x, \cos x)$, $(\sin^3 x, 0)$, where $0 < x < \frac{\pi}{2}$.

a) If $A(x)$ represents the area of the triangle, write the formula for $A(x)$.

b) Find the value of $x$ for which $A(x)$ is a *maximum*. Justify, of course.

c) What is the permeter of triangle when the area is at its maximum?

**XX** Widgets-R-Us has a revenue function $R(x) = 52500x + 450x^2 - x^3$. What production level will yield a maximum revenue?

**XXI** A lighthouse is on an island one mile from a straight shoreline. Suppose that the light is turning at the rate of two revolutions per minute in the counterclockwise direction.

a) How fast is the light beam moving along the shore at the instant it makes an angle of 45° with the shoreline.

b) How fast is the light beam moving at the instant the angle is 90°?

**XXII** Each of the following functions has interesting but different behavior at $x = 2$. Graph each function (You may copy the picture from your grapher if you provide the window settings.) Tell at, $x = 2$, whether the function (list all that apply)

| | | |
|---|---|---|
| has a function value | has a tangent line | has a cusp |
| has a limit | is differentiable | has extreme value |
| is continuous | has an asymptote | is concave up |
| is concave down | is increasing | is decreasxing |

1) $f(x) = (x-2)^2 + 1$

2) $f(x) = (x-2)^3 + 1$

3) $f(x) = (x-2)^4 + 1$

4) $f(x) = (x-2)^{\frac{1}{3}} + 1$

5) $f(x) = (x-2)^{\frac{2}{3}} + 1$

6) $f(x) = (x-2)^{\frac{1}{2}} + 1$

**XXIII** Amos Take analyzes $f(x) = x^2$, defined for all $x$ in the closed interval $[-2, 3]$. He finds that $f''(-2) = 2$ and concludes that the graph is concave upward there, and thus that there is a local minimum at $x = -2$. Explain to Amos the error in his thinking.

**XXIV** A rectangle is inscribed in the region bounded by the $x$-axis and the parabola $y = 36 - x^2$. The base of rectangle is on the $x$-axis.

a) Find the length and width of the rectangle of greatest area.

b) Find the length and width of the rectangle of greatest perimeter.

c) Why does the rectangle of greatest area have the greatest perimeter?

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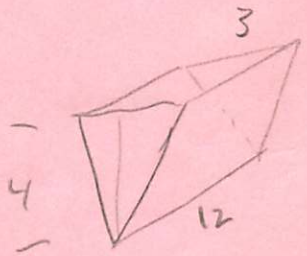
#




$V = \frac{1}{2} b h \ell$

$V = \frac{1}{2} 3 \cdot 4 \cdot 12$

$= 72$ total capacity.

$V = \frac{1}{2} \cdot \frac{3}{2} \cdot x \cdot 6$

$= 18$ cu ft so far

54 more cu ft @ 3 cu ft/in.

18 min.

∴ tough fuel at 2:38

$\frac{x}{2} = \frac{3}{4}$

$x = \frac{3}{2}$

$\frac{x}{h} = \frac{3}{4}$

$x = \frac{3}{4} h$

b) $\frac{dh}{dt} = ?$

$\frac{dV}{dt} = 8$

$V = \frac{1}{2} x \cdot h \cdot 12$

$= \frac{1}{2} \frac{3}{4} h \cdot h \cdot 12$

$V = \frac{9}{2} h^2$

$\frac{dV}{dt} = 9h \frac{dh}{dt}$

$8 = 9(4) \frac{dh}{dt}$

$\frac{1}{12} = \frac{dh}{dt}$

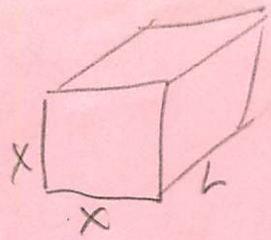
The height is rising at 1" per minute.

c) $\left.\frac{dh}{dt}\right| = ?$ 2 ers.

2:40

The water is over flowing now and the height is not changing.

III




$4x+L=72$

$L=72-4x$

$V=lwh$

$V(x)=(72-4x)x^2$

$=72x^2-4x^3$

$V'(x)=144x-12x^2$

$12x(12-x)$

$x=0$ reject $x=12$

+ 0 −

0 12

max f' test.

max volume is

$(144)(72-48)$

$(144)(24)=$

3456

Changed Domain to $[-2,0]$

IV

$f(x)=4x^3-12x+7$ $[0,2]$

$12x^2-12 \overset{set}{=} 0$

$12(x-1)(x+1)=0$

− 0 +

−2 2

$\{7$

$-24$

(0, 7) max ENDPT def'n of it.

(1, −1) min f' test.

(2, 15) max ENDPT inc to it.

V

$y=x^3-6x^2+3x+5$

$y'=3x^2-12x+3$

$y''=6x-12$

$3x^2-12x+3 \overset{set}{=} 6x-12$

$3x^2-18x+15=0$

$3(x^2-6x+5)=0$

$3(x-5)(x-1)=0$

$x=\{1,5\}$

| | 1 | −6 | 3 | 5 |
|---|---|---|---|---|
| 1 | 1 | −5 | −2 | 3 |
| 5 | 5 | −1 | −2 | −5 |

The candidate and the slope are changing at the same rate when $x=1$ and when $x=5$.

(1, 3) (5, −5)

$$1100 + (t-17)^2 \leq 1400 - \frac{1000}{t+7}$$

$$0 = 300 - (t-17)^2 - \frac{1000}{t+7}$$

$$\frac{300t + 2100 - (t^2 - 34t + 289)(t+7) - 1000}{t+7}$$

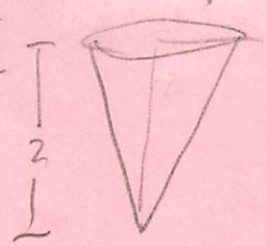
$$-1000$$
$$300t + 2100 - t^3 + 34t^2 - 289t$$
$$-7t^2$$
$$289t - 2023$$
$$238t - 3100$$

$$-t^3 + 27t^2 + 249t - 5123 - 923$$

$$-\left(\frac{923 + 249t - 27t^2 - t^3}{t+7}\right)$$

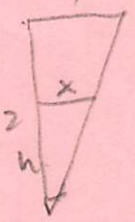
$$33.59 \quad 2.88 \quad -9.49$$

VI




$V = \frac{1}{3}\pi r^2 h$ $\quad \frac{dh}{dt}\Big|_{=?}$ $\frac{1}{8}$ full

$\frac{dr}{dt} = 2$




$\frac{x}{h} = \frac{1}{2}$

$2x = h$

$x = \frac{h}{2}$

full vol $V = \frac{1}{3}\pi 1^2 2$

$= \frac{2\pi}{3}$ cu ft

$\frac{1}{8}$ full $= \frac{\pi}{12}$ cu ft

$V = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h$ $\quad \frac{dh}{dt} = \frac{8}{\pi}$

$\frac{\pi}{12}h^3 \overset{\text{set}}{=} \frac{\pi}{12}$

$\therefore h = 1$

$V = \frac{\pi}{12}h^3$

$\frac{dV}{dt} = \frac{\pi}{4}h^2\frac{dh}{dt}$

$2 = \frac{\pi}{4}(1)^2\frac{dh}{dt}$

The height is rising at the rate of $\frac{8}{\pi}$ feet/minute.

VII $\quad V(t) = 1100 + (t-17)^2$

$w(t) = 1400 + 100(t+7)^{-1}$

$w'(t) = \frac{-1000}{(t+7)^{-2}}$

a) $v'(t) = 2(t-17) \overset{\text{set}}{=} 0$

$t = 17$

$v'' = 2$

min at 17 by 2nd d test.

the max for wood is 1400 but they never achieve that.

b) $V(16) = 1100 + 1^2$

$\approx 1101$

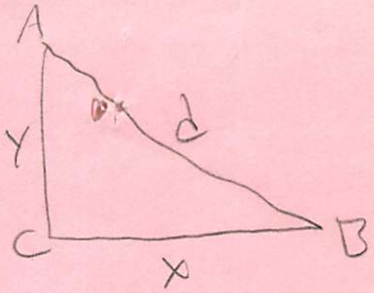
$w(16) = 1400 - \frac{1000}{23}$

$\approx 1356$

c) They are the same during the 82-83 year $\approx 1299$ students

the 2013-14 year $\approx 1375$ students.

VIII




$\frac{dy}{dt} = 2$ $\quad \frac{dx}{dt} = -3$ $\quad \frac{dd}{dt}\Big| = ?$ $\quad x=6,\ y=8$

$d^2 = x^2 + y^2$

$2d\frac{dd}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$

$10\frac{dd}{dt} = 6(-3) + 8(2)$

$= \frac{-18+16}{10}$

$= \frac{-2}{10} = -\frac{1}{5}$

$\frac{dd}{dt}$

The hypot is getting shorted at $\frac{1}{5}$" per second

$d^2 = 36 + 64$

$d = 10$

$\frac{dA}{dt}\Big| = ?$

$A = \frac{1}{2} x y$

$\frac{dA}{dt} = \frac{1}{2} y \frac{dx}{dt} + \frac{1}{2} x \frac{dy}{dt}$

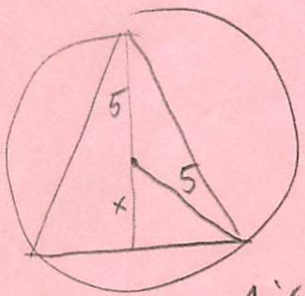
$= \frac{1}{2}(8)(-3) + \frac{1}{2}(6)(2)$

$-12 + 6$

$= -6$

The area is getting smaller at 6 sq" per sec

IX




$A = \frac{1}{2} b h$

$A(x) = \frac{1}{2}(x+5)\sqrt{25-x^2}$

$A'(x) = (x+5)\left(\frac{-2x}{2\sqrt{25-x^2}}\right) + \sqrt{25-x^2}$

$A'(x) = \frac{-x^2 - 5x + 25 - x^2}{\sqrt{25-x^2}}$

$= \frac{-(2x^2 + 5x - 25)}{\sqrt{25-x^2}}$

$-(2x-5)(x+5)$

$+ \quad 0 \quad -$

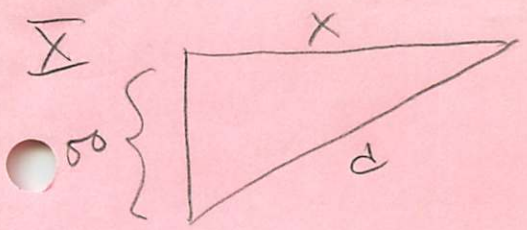
$0 \qquad \frac{5}{2}$

max at $x = \frac{5}{2}$ f'tal.

Altitude is

$\frac{5}{2} + 5 = \frac{15}{2} = 7\frac{1}{2}$.

X




$\frac{dd}{dt} = 20$

$\frac{dx}{dt} = ?$

$d^2 = x^2 + 50^2$

$2d\frac{dd}{dt} = 2x\frac{dx}{dt}$

$50^2 + x^2 = 130^2$

$2500 + x^2 = 16900$

$14400$

$x = 120$

$\frac{65}{130} = \frac{dx}{dt}$

$120$

$65$

The wind is blowing at $21\frac{2}{3}$ ft/sec.

XI

$A(t) = \frac{3t}{t+1}$

$B(t) = \frac{t^2}{t+1} + 1$

a) $A(1) = \frac{3}{2}$ billion

$B(1) = \frac{1}{2} + 1 = \frac{3}{2}$ billion.

b) Cannot tell both could have the same assets in 1 year.

c) $A' = \frac{(t+1)3 - 3t}{(t+1)^2}$

$A' = \frac{3}{(t+1)^2}$

$B' = \frac{(t+1)^2 2t - t^2}{(t+1)^2}$

$B' = \frac{t^2 + 2t}{(t+1)^2}$

d) $A'(1) = \frac{3}{4}$

$= \frac{1+2}{4}$

e) No Both are growing at the same rate

f) $A'' = \frac{-3(2)}{(t+1)^3}$

$\frac{-6}{(t+1)^3}$

$B'' = \frac{(t+1)^2(2t+2) - (t^2+2t)\cdot 2(t+1)}{(t+1)^4}$

$B'' = \frac{2}{(t+1)^3}$

g) $A''(1) = \frac{-6}{8} = -\frac{3}{4}$

$B''(1) = \frac{2}{8} = \frac{1}{4}$

Next page.

XII $x = t + \frac{1}{t}$ $y = t - \frac{1}{t}$

$\frac{dx}{dt} = 1 - \frac{1}{t^2}$ $\frac{dy}{dt} = 1 + \frac{1}{t^2}$

$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$ chain Rule.

a) $\frac{dy}{dx} = \frac{\frac{t^2+1}{t}}{\frac{t^2-1}{t}} = \frac{t^2+1}{t^2-1}$ ✓

b) $\frac{d\left(\frac{dy}{dx}\right)}{dx} = \frac{d\left(\frac{t^2+1}{t^2-1}\right)}{dt} \cdot \frac{dt}{dx}$

$\frac{d^2y}{dx^2} = \frac{(t^2-1)2t - (t^2+1)2t}{(t^2-1)^2} \cdot \left(\frac{t^2}{t^2-1}\right)$

$= \frac{2t^3 - 2t - 2t^3 - 2t}{(t^2-1)^2} \cdot \frac{t^2}{t^2-1}$

$= \frac{-4t^3}{(t^2-1)^3}$

c)

$x + y = 2t$

$x - y = \frac{2}{t}$

$x - y = \frac{2}{\frac{x+y}{2}}$

$x^2 - y^2 = 4$

$x^2 - y^2 = 4$ hyperbola

XIII let $x =$ one non Neg #

$1 - x =$ other non Neg #

a) $(1-x)^2 + x^3 = f(x)$

$f'(x) = 3x^2 + 2(1-x)(-1)$

$= 3x^2 + 2x - 2$

$f''(x) = 6x + 2$

$x = \frac{-2 \pm \sqrt{4+24}}{6}$

$\frac{-2 \pm 2\sqrt{7}}{6} - \frac{1 \pm \sqrt{7}}{3}$

$f''\left(\frac{-1+\sqrt{7}}{3}\right) = 6\left(\frac{-1+\sqrt{7}}{3}\right) + 2$

$-2 + 6\sqrt{7} + 2$

$6\sqrt{7} > 0$

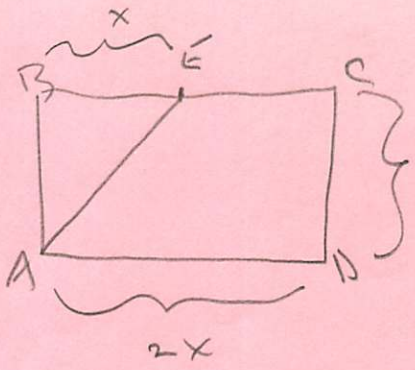
$\therefore$ min at $\frac{-1+\sqrt{7}}{3}$

$\frac{-1-\sqrt{7}}{3}$ is reject since both Numbers are "non Negative"

$f'(0) = -2 \therefore < 0$ So fis decreasing from $x=0$ il max ~~first~~

The two numbers are 0, 1.

XIV




$\rightarrow \frac{10}{2x} = \frac{5}{x}$ $\sqrt{\left(\frac{5}{x}\right)^2 + x^2} = f(x)$

$f' = \sqrt{\frac{25+x^4}{x^2}} \cdot \frac{4}{5}$

$f'(x) = \frac{1}{2\sqrt{\frac{25+x^4}{x^2}}} \cdot \left[\frac{x^2(4x^3) - (25+x^4)2x}{x^4}\right]$

$\frac{4x^5 - 50x - 2x^5}{x^4}$

$\frac{2\sqrt{25+x^4}}{x}$ $\frac{2x^4 - 50}{x^3}$

$\frac{x}{\sqrt{25+x^4}} \cdot \frac{2(x^4-25)}{x^3 2}$

$f' = \frac{x^4-25}{\sqrt{25+x^4}} \overset{\text{set}}{=} 0$ $x = \sqrt[4]{25}$

$x = \sqrt{5}$

$-\ 0\ +$

$\sqrt{5}$

min at $x = \sqrt{5}$

$f'$ test.

Dimensions are

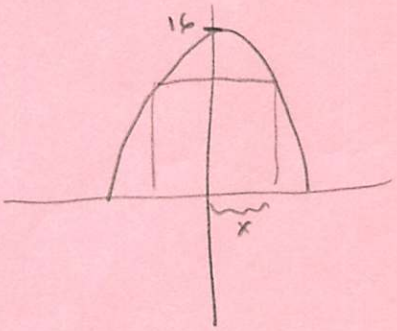
$2x \times \frac{10}{2x}$

$2\sqrt{5} \times \frac{10}{10}\sqrt{5}$

$2\sqrt{5} \times \sqrt{5}$ it's a square!

XV

$y = 16 - x^2$




a)

$A = 2xy$

$A = 2x(16 - x^2)$

$A(x) = 32x - 2x^3$

$A'(x) = 32 - 6x^2 \overset{set}{=} 0$

$x^2 = \frac{16}{3}$ $\quad \frac{4}{\sqrt{3}}$

$A''(x) = -12x$ $\quad x = \frac{4\sqrt{3}}{3}$ $\quad 16 - \frac{16}{3} = \frac{32}{3}$

$A''\left(\frac{4\sqrt{3}}{3}\right) < 0$

$\therefore$ max f'' test, Dimensions $\frac{8\sqrt{3}}{3} \times \frac{32}{3}$

b) $P(x) = 4x + 2y$

$= 4x + 2(16 - x^2)$

$4x + 32 - 2x^2$

$P' = 4 - 4x$

$= 4(1 - x) \overset{set}{=} 0$

$x = 1$

$P'' = -8x \Big| < 0 \therefore$ max at $x = 1$

$x = 1$

Dimensions

$2 \times 15$

XVI $C(x) = 800 + .4x + .0002x^2$

a) $\bar{C}(x) = \frac{800}{x} + .4 + .0002x$

$\bar{c}' = -\frac{800}{x^2} + .0002 \overset{\text{set}}{=} 0$

$x^2 = 4{,}000{,}000$

$x = 2000$

$\bar{c}'' = \frac{1600}{x^3}$

① $c''(2000) > 0 \therefore$ min at $x = 2000$ f "cup.

② $\bar{C}(x)\Big|_{x=4000} = \frac{800}{4000} + .4 + .0002(4000)$

$.2 + .4 + .8 = 1.4$

$\bar{c}(x)\Big|_{x=2000} = 1.2$

$\bar{c}(x)\Big|_{x=1000} = 1.4.$

B) $C(x) = x^3 - 6x^2 + 13x$

$\bar{C}(x) = x^2 - 6x + 13$

$\bar{c}' = 2x - 6 \overset{\text{set}}{=} 0$

$x = 3$

$\bar{c}'' = 2 \therefore$ min at $x = 3$ f "cup.

$\bar{C}(3) = 9 - 18 + 13$

$4$

$C'(x) =$ marginal cost

$c'(x) = 3x^2 - 12x + 13$

$c'(3) = 27 - 36 + 13$

$= 4$

4 marginal cost at $x = 3$

XVI continued

c) $C(x) = 2x^2 + 5x + 18$

$\bar{C}(x) = 2x + 5 + \frac{18}{x}$

$\bar{C}' = 2 - \frac{18}{x^2} \overset{\text{set}}{=} 0$

$x^2 = 9$

$x = 3$

$\bar{C}'' = \frac{36}{x^3} > 0 \therefore$ min at $x = 3$ f'' test.

$\bar{C}(3) = 2(3) + 5 + \frac{18}{3}$

$6 + 5 + 6 = 17$

$C'(x) = 4x + 5$

$C'(3) = 12 + 5$

$17$

D) WTS

$A_{\text{average}} = \frac{C(x)}{x}$ $\frac{u}{v}$

$A' = \frac{x C'(x) - C(x)}{x^2} \overset{\text{set}}{=} 0$

$C'(x) = \frac{C(x)}{x}$

marginal = average

$x C'(x) - C(x) = 0$

$x C'(x) = C(x)$

$x = \frac{C(x)}{C'(x)}$

Unify - - - -

$A' = \frac{x C'(x) - C(x)}{x^2}$

$= \frac{C'(x) - \frac{C(x)}{x}}{x}$

we know $x > 0$

If $x \to \infty$ then $\frac{C(x)}{x} \to 0$

so for large $x$ we have $\frac{+}{+} = +$

so for $0 < x < \frac{C(x)}{C'(x)}$ the number line is $-$. (signs alternate)

so we have a min by f' test.

0 (+)

0 $x = \frac{C(x)}{C'(x)}$

XVI

d) $\bar{C}(x) = \frac{C(x)}{x}$

$\bar{C}'(x) = \frac{x C'(x) - C(x)}{x^2} \overset{\text{set}}{=} 0$

$x C'(x) - C(x) = 0$

$x C'(x) = C(x)$

$C'(x) = \frac{C(x)}{x}$

Marginal cost = Ave cost but ....

do we have a min or max?

Solve for x ...

$x = \frac{C(x)}{C'(x)}$

$\bar{C}'(x) = \frac{x C'(x) - C(x)}{x^2}$

If $\bar{C}''\left(\frac{C(x)}{C'(x)}\right) > 0$

Then we have found a min.

$\bar{C}''$ $\bar{C}''(x) = \frac{x^2\left[C'(x) + xC''(x) - C'(x)\right] - \left(xC'(x) - C(x)\right)2x}{x^4}$

$= \frac{x^3 C''(x) - 2x^2 C'(x) - 2xC(x)}{x^4}$

$= \frac{x^2 C''(x) - 2x C'(x) - 2C(x)}{x^3}$

$\bar{C}''\left(\frac{C(x)}{C'(x)}\right) = \frac{\left(\frac{C(x)}{C'(x)}\right)^2 C''(x) - 2\left(\frac{C(x)}{C'(x)}\right)C'(x) - 2C(x)}{\left(\frac{C(x)}{C'(x)}\right)^3}$

$= \frac{C''(x)}{\frac{C(x)}{C'(x)}}$

$\left(\bar{C}\right)'' = \frac{C''(x)\,C'(x)}{C(x)}$

$$\bar{C}'' = \frac{C''(x)C'(x) - C'(x)}{C(x)^2} = \frac{1}{x}$$

$$\bar{C}'' = \frac{C''(x)}{x}$$ so $\bar{C}''$ has same signs as $C''$

If $C$ has a min then $C'' > 0$ and then so is $\bar{C}''(x)$

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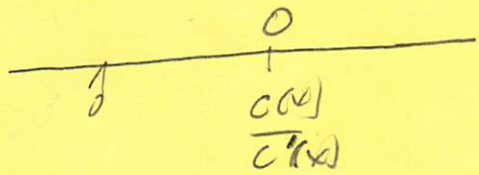
$$\bar{C}'(x) = \frac{xC'(x) - C(x)}{x^2}$$

$xC'(x) - C(x)$



XVII

$x = \frac{2500}{q^2}$ $\quad 2500 = \frac{2500}{q^2}$

$q = \frac{50}{\sqrt{x}}$ $\quad \therefore q = 1$

Price per unit is $1 dollar.

$R(x) = 50\sqrt{x}$

$P(x) = 50\sqrt{x} - (.5x + 500)$

$P'(x) = \frac{25}{\sqrt{x}} - .5 \overset{\text{set}}{=} 0$

$.5\sqrt{x} = 25$

$P''(x) = \frac{-25}{2x^{3/2}} < 0$ $\quad .25x = 25^2$ $\quad \frac{100}{25}$

$\therefore$ max at critical value $x = 2500$.

XVIII

$q = 90 \qquad C = 60$

Let $x$ = # of bippies

$(90 - \frac{1}{10}(x - 100))$ Price per bippie (if more than 100 ordered.)

$P = x(90 - \frac{1}{10}(x-100)) - 60x$

$= 90x - \frac{1}{10}x^2 + 10x - 60x$

$= 40x - \frac{1}{10}x^2$

$P' = 40 - \frac{1}{5}x \overset{\text{set}}{=} 0$

$x = 200$

$P'' = -\frac{1}{5} < 0 \therefore$ max at $x = 200$

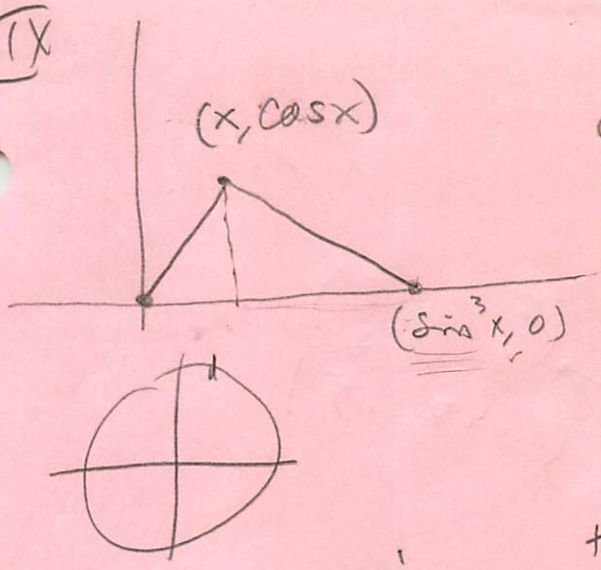
$P(400) = 400\left(90 - \frac{1}{10}(300)\right) - 60 \cdot 400$

$400(60) - 400 \cdot 60$

$= 0$

The discount would eat up all profit made.

XIX




$A = \frac{1}{2}bh$

a) $A(x) = \frac{1}{2}\underset{u}{\sin^3 x}\ \underset{v}{\cos x}$

b) $A' = \frac{3}{2}\cos^2 x \sin^2 x - \frac{1}{2}\sin^4 x$

$= \frac{1}{2}\sin^2 x\left(3(1-\sin^2 x) - \sin^2 x\right)$

$= \frac{1}{2}\sin^2 x\left(3 - 4\sin^2 x\right) \overset{\text{set}}{=} 0$

$\sin^2 x = \frac{3}{4}$

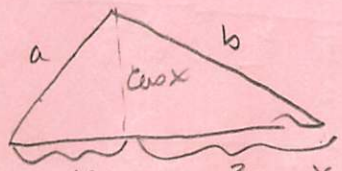
$\sin x = \frac{\sqrt{3}}{2}$

$x = \frac{\pi}{3}$

$A'$: + 0 − on $0, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}$

max at $x = \frac{\pi}{3}$ f' test. $A'\left(\frac{\pi}{4}\right) = +\left(3 - 4\cdot\frac{1}{4}\right)$ +

c) Perimeter?




$\frac{3\sqrt{3}}{8} - \frac{\pi}{3}$

$a^2 = \cos^2 x + x^2$

$a = \sqrt{\frac{1}{4} + \frac{\pi^2}{9}}$

NO See back Side of This Page

$b^2 = \cos^2 x + (\sin^3 x - x)^2$

$b = \sqrt{\frac{1}{4} + \left(\frac{3\sqrt{3}}{8} - \frac{\pi}{3}\right)^2}$

Perimeter: $\sqrt{\frac{9+4\pi^2}{36}} + \sqrt{\frac{1}{4} + \left(\frac{3\sqrt{3}}{8} - \frac{\pi}{3}\right)^2} + \frac{3\sqrt{3}}{8}$

$\sqrt{9+4\pi^2}$ .639 $\sin^3 x$

6 .650

1.160 2.449

XX

$R(x) = 52500x + 450x^2 - x^3$

$R'(x) = 52500 + 900x - 3x^2$

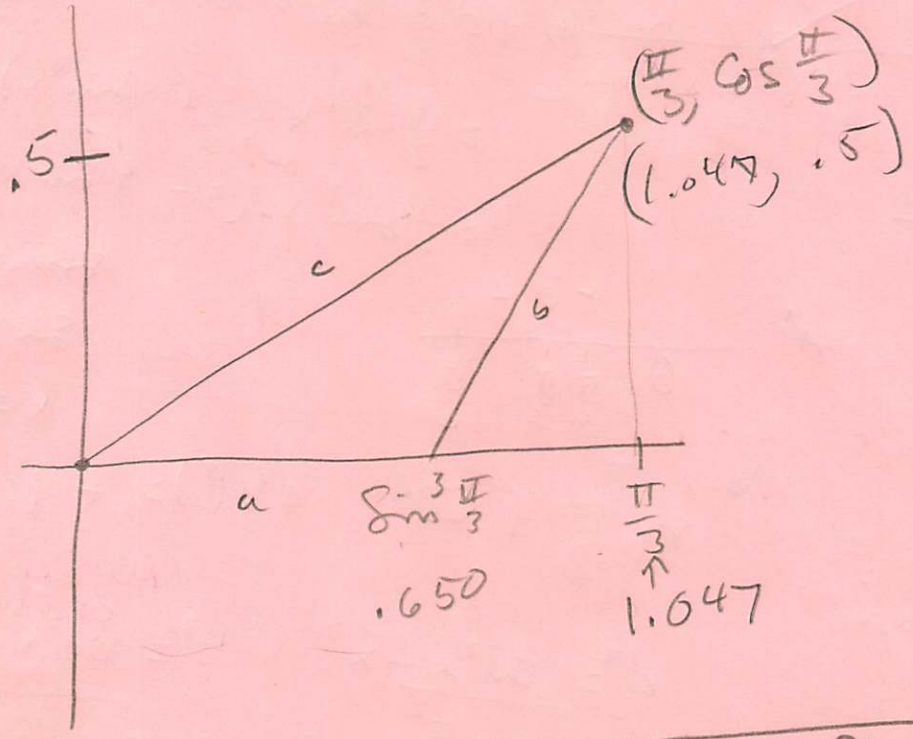
$-3(17500 + 300x - x^2) \overset{\text{set}}{=} 0$

$-3(x - 350)(x + 50)$

$R'' = 900 - 6x$

$R''(350) < 0$ ∴ max f'' test.

The max revenue occurs when 350 units are produced.




$a = .650$

$$b = \sqrt{\left(\frac{\pi}{3} - \sin\frac{\pi}{3}\right)^2 + \cos^2\frac{\pi}{3}}$$

$$\sqrt{.158 \quad + .25}$$

$$\sqrt{.408}$$

$$= .639$$

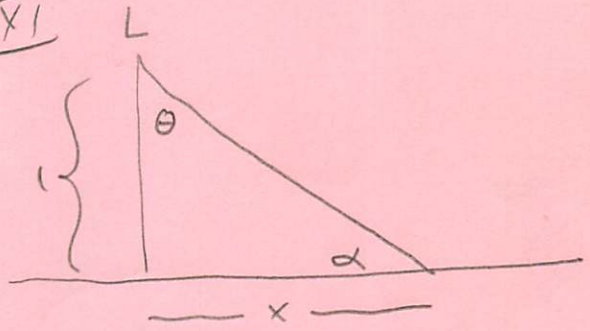
$$c = \sqrt{\frac{\pi}{3}^2 + \cos^2\frac{\pi}{3}}$$

$$= \sqrt{1.347}$$

$$= 1.160$$

$$a + b + c = 2.449$$

XXI




$\frac{dx}{dt} = ?$

2 Rev = min

$4\pi$ rad/min $= \frac{d\theta}{dt}$

$\tan\theta = x$

$\sec^2\theta \frac{d\theta}{dt} = \frac{dx}{dt}$

when $\alpha = 45^\circ$ $\theta = \frac{\pi}{4}$

a) $2 \cdot 4\pi =$

$8\pi = \frac{dx}{dt}$

b) when $\alpha = 90$, $\theta = 0$

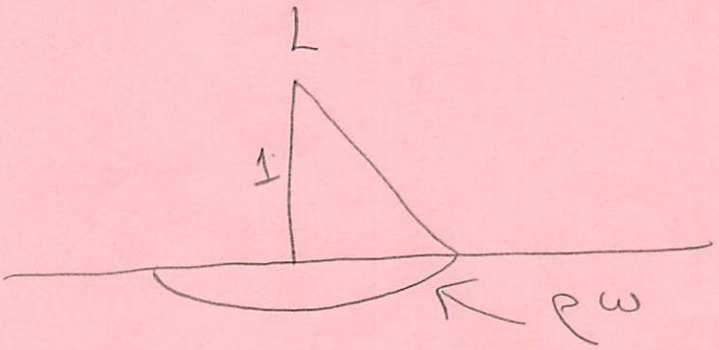
$\sec^2\theta \frac{d\theta}{dt} = \frac{dx}{dt}$

$\theta = 0^\circ$

$\alpha = 90^\circ$ $1 \cdot 4\pi =$

$4\pi > \frac{dx}{dt}$.

spot of light moves $\frac{1}{2}$ as fast.

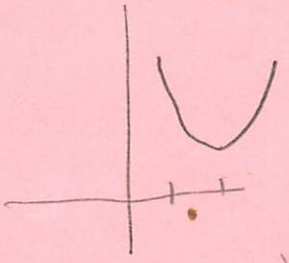



Longer radius ∴ velocity is greater

So we expect velocity to be less when the light

XXII

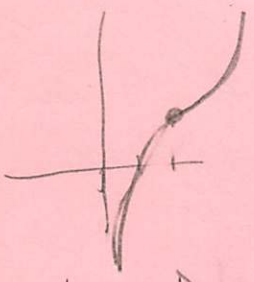
1) $f(x) = (x-2)^2+1$



a, b, c, D, h

NO e, f, g

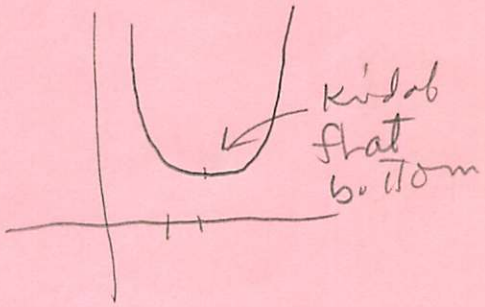
2) $f(x) = (x-2)^3+1$



a, b, c, D,

No e, ~~d~~, f, g

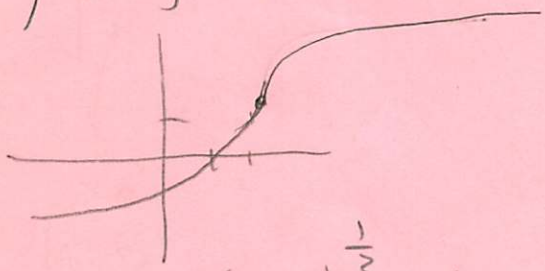
3) $f(x) = (x-2)^4+1$




a, b, c, D, h

No e, f, g

4)

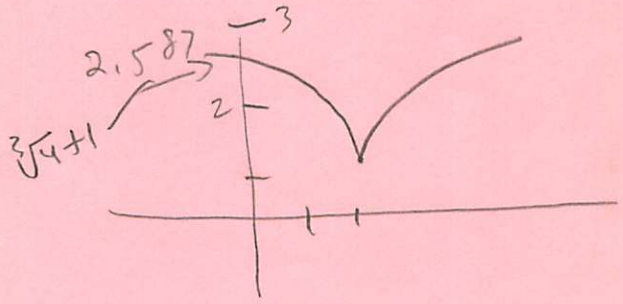


$f(x) = (x-2)^{\frac{1}{3}}+1$

a, b, c,

No d, e, f, g, h

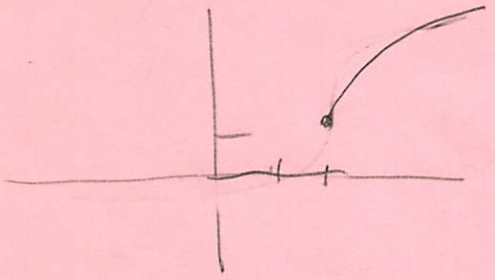
5) $f(x) = (x-2)^{\frac{2}{3}}+1$




a, b, c, g h

No d e f

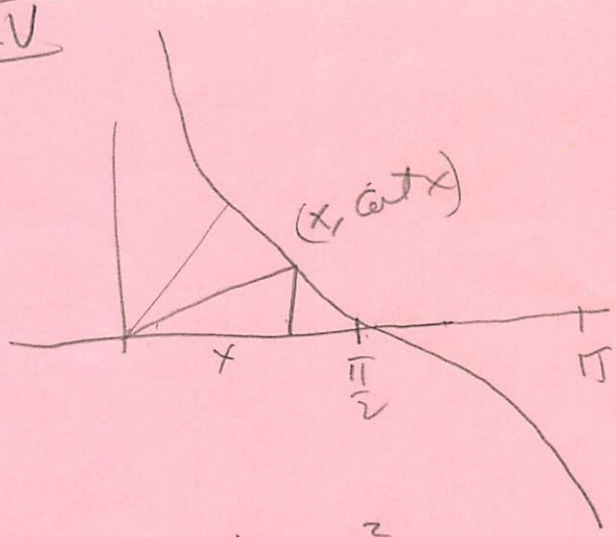
6) $f(x) = (x-2)^{\frac{1}{2}}+1$



a, b, c

No d e f g h

x x V




$A = \frac{1}{2} x \cot x$

$\lim_{x \to 0} A = \left(\frac{1}{2} \cdot \frac{x}{\sin x} \cdot \cos x\right)$

$= \frac{1}{2} \cdot 1 \cdot 1 = \frac{1}{2}$

$\frac{1}{2}\cot x - \frac{1}{2} x \csc^2 x$

$\frac{1}{2}\sin x\left(\cos x - \frac{x}{\sin x}\right)$

$s^2 + c^2 = 1$

$1 + \cot^2 = \csc^2$

$\frac{1}{2}\left[\cot x - 1 - x\cot^2 x\right] \overset{\text{set}}{=} 0$

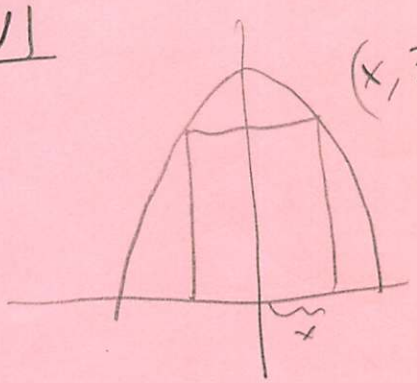
$\lim_{x \to 0} \frac{x}{\sin x} = 1 \quad \therefore \quad x = 0$

$\lim_{x \to 0} \frac{1}{2} \cos x \cdot \frac{x}{\sin x}$

$\frac{1}{2} \cdot 1 \cdot 1 = \frac{1}{2}$ ✓

The max area is $\frac{1}{2}$ but is never achievable.

XXVI




$A(x) = 2x(36-x^2)$

$= 72x - 2x^3$

$A' = 72 - 6x^2$

$6(12-x^2) \overset{\text{Set}}{=} 0$

$x = 2\sqrt{3}$

$A'' = -12x \Big|_{x=2\sqrt{3}} < 0$ ∴ max f'' test

a) $4\sqrt{3} \times 24$ 166.28 area

b) $2 \times 35$ 70 area.

c) it doesn't.

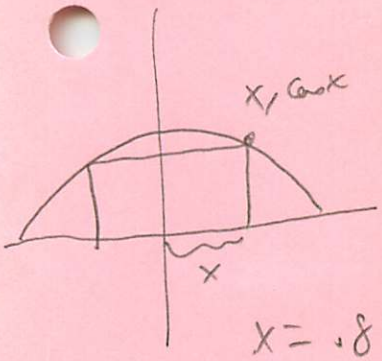
$P = 4x + 72 - 2x^2$

$P' = 4 - 4x$

$x = 1$

$P'' = -4$ ∴ max f'' test.

XXIII




$A = lw$

$A = 2x \cos x$ (u, v)

$A' = 2\cos x - 2x \sin x \overset{\text{set}}{=} 0$

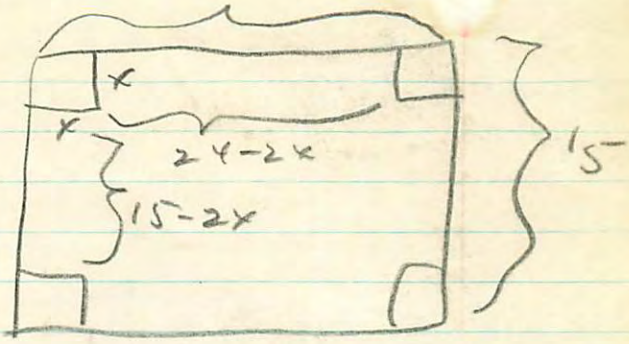
$x = \cot x$

$x = .8603$

XXIV $f''$ test requires used of critical value of x value on ind PT.

XXV

Eg 1 P. 158




$V = hLW$

$V(x) = x(24-2x)(15-2x)$

$V(x) = 4x^3 - 78x^2 + 360x$

What are the restrictions on x? well x must be positive but less than $\frac{15}{2}$. In order to simplify and generalize our method we allow a "zero" dimension. Thus far we have the restrictions on the domain

$$0 \leq x \leq \frac{15}{2}$$

Since V is a polynomial fctn we can find V', we know that $V' \stackrel{\text{set}}{=} 0$ will yield critical points of V which could be the maximum we are seeking.

$V'(x) = 12x^2 - 156x + 360$

$= 12(x^2 - 13x + 30)$

$12(x-10)(x-3) \stackrel{\text{set}}{=} 0$

$x = 10 \quad x = 3$ Notice that $x = 10$ is beyond the domain of x so we consider $x = 3$.

$V''(x) = 24x - 156$

$V''(3) = 72 - 156 < 0$ ∴ 3 is a relative max.

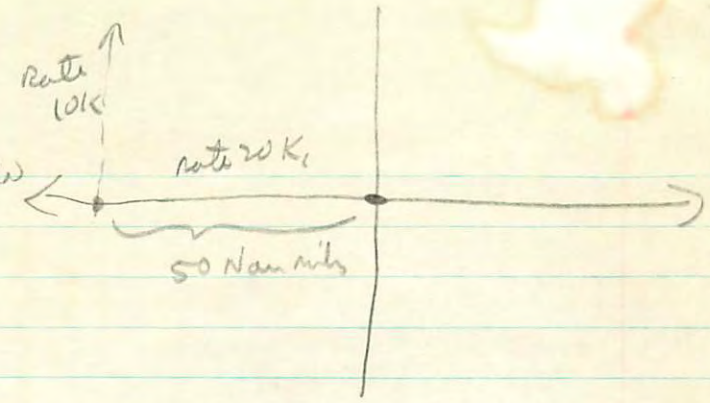
Question is 3 the absolute max? Since we are working within a closed interval we must consider the endpts to check to see if we have an absolute max.

$f(0) = 0 \quad f(\frac{15}{2}) = 0$ obviously not a max

∴ $3 = x$ is our max.

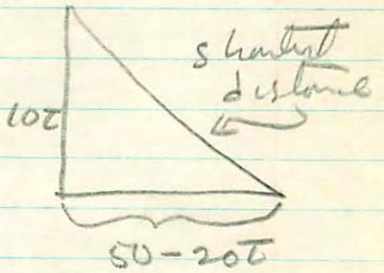
Answer the original Question. 3, 18, 9

eg 2 pg 160




$c^2 = a^2 + b^2$

| Ship | Rate | time | dist in his direction |
|---|---|---|---|
| W | 20 | $t$ | $20t$ |
| N | 10 | $t$ | $10t$ |




Let $d^2 = f(t)$

$f(t) = (10t)^2 + (50 - 20t)^2$

$= 100t^2 + 2500 - 2000t + 400t^2$

$= 500t^2 - 2000t + 2500$

$f(t) = 500(t^2 - 4t + 5) \quad t \geq 0$

$f'(t) = 500(2t - 4)$

$500(2t - 4) \overset{\text{Set}}{=} 0$

$t = 2$

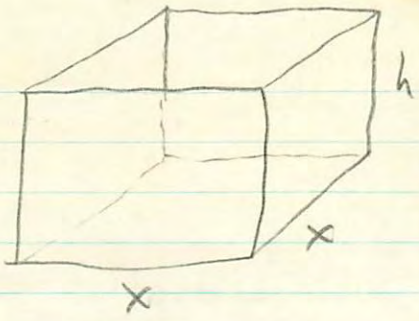
$f''(t) = 500(\quad) > 0$ ∴ 2 is a minimum.

Since $f(0)$ (an end pt.) $= 2500$

and $f(2) = 500$ $f(2)$ is the absolute min.

thus the answer is the ships will be closest at 2 pm.

eg 3 pg 161




the problem is one of finding a minimum that minimum is in reference to total surface area.
that surface area is the base $x^2$ and the four sides each of area $xh$.
the volume of the box is $x^2h = 32$
ie $h = \frac{32}{x^2}$

thus $A = x^2 + 4hx$ reduces to
$A = x^2 + \frac{4 \cdot 32}{x}$

$A = x^2 + \frac{128}{x}$ restriction $x > 0$ (Note Not $x \geq 0$ since $x$ in denominator)

$A = x^2 + 128x^{-1} = f(x)$

$\frac{dA}{dx}$

$f'(x) = 2x - 128x^{-2}$

$= 2x - \frac{128}{x^2} \overset{\text{Set}}{=} 0$

$x^3 - 64 = 0$

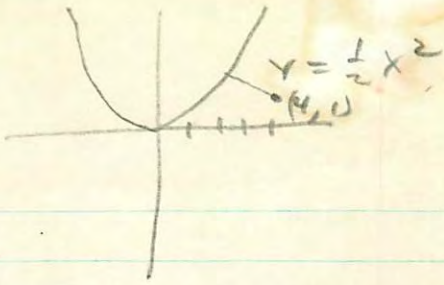
$x^3 = 64$

$x = 4$ thus only one pt to consider since $f(x)$ is everywhere defined for the Domain.

$f''(x) = 2 + 256x^{-3}$

$f''(4) = \quad > 0 \therefore$ minimum.

thus the required dimensions are 4 ft by 4 FT. and 2 ft. (the "h").




Eg 4 pg 162

we want to find a minimum distance.

$P_1(x, y)$ on $y = \frac{1}{2}x^2$ and $P_2(4, 1)$

the distance formula is $d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$

$d^2 = (x-4)^2 + (y-1)^2$ we want only one variable in our fcn so we substitute $y = \frac{1}{2}x^2$

$d^2 = (x-4)^2 + \left(\frac{x^2}{2} - 1\right)^2$

$d^2 = x^2 - 8x + 16 + \frac{x^4}{4} - x^2 + 1$

$d^2 = \frac{x^4}{4} - 8x + 17$ Restrictions on domain? None.

$f(x) = \frac{x^4}{4} - 8x + 17$

$f'(x) = x^3 - 8 \stackrel{\text{set}}{=} 0$

$x = 2$

$f''(x) = 3x^2 > 0$ $\therefore$ $x = 2$ is minimum

f is defined everywhere & has no endpts
$\therefore$ since $x = 2$ is only min it is absolute min.

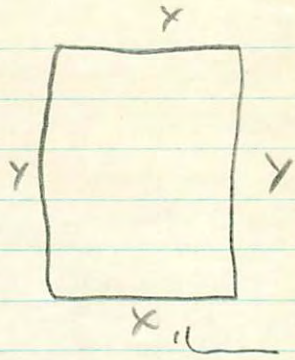
the coordinates asked for then

$y = \frac{1}{2}(2)^2$

$y = \frac{1}{2}4$

$y = 2$ $(2, 2)$

Oral ex pg 163




① $C = 4x + 2y$
Cost for fence if one of side x is the $4/ft. material is
$C = 3x + x + y + y$

② $xy = 200$
area in terms of feet

③ $C = f(x) = 4 + \frac{400}{x}$

Substituting for y ($xy = 200 \therefore y = \frac{200}{x}$) we have cost in terms of x only.

④ $f'(x) = 4 - \frac{400}{x^2}$ derivative of #3 above.

⑤ $f'(x) = 0$ if $x = 10$ set $f'(x) = 0$ & solve.

⑥ $f''(10) > 0$ means we have a minimum

⑦ $y = 20$ subst in #2 above using $x = 10$

⑧ $C = 80$ the minimum cost of the fence.

① Product is max

$x(12-x) = f(x) \quad 0 \le x \le 12$

$12x - x^2 = f(x)$

$12 - 2x \overset{\text{Set}}{=} 0$

$2x = 12$

$x = 6$

| | |
|---|---|
| $x=0$ | $f(0)=0$ |
| $x=6$ | $f(6)=36$ ← max |
| $x=12$ | $f(12)=0$ |

$f''(x) = -2 < 0 \therefore$ max.

The Numbers are 6 and 6

② $x^2 + (12-x)^2 = f(x) \qquad 0 \le x \le 12$

$x^2 + 144 - 24x + x^2 = f(x)$

$2x^2 - 24x + 144 = f(x)$

$2(x^2 - 12x + 72) = f(x)$

$f'(x) = 2(2x - 12) \overset{\text{Set}}{=} 0$

$2x = 12$

$x = 6$

$f''(x) = 4 > 0$ min

$f(0) = 144$

$f(12) = 144 \therefore f(6)$ is min.

The Numbers are 6 and 6

(3)

$x^2(12-x) = f(x) \quad 0 \le x \le 12$

$12x^2 - x^3 = f(x)$

$24x - 3x^2 = f'(x)$

$3x(8-x) \overset{\text{set}}{=} 0$

$x = 0 \quad x = 8$

$f''(x) = 24 - 6x$

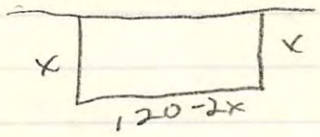
$f''(0) = > 0$ min

$f''(8) = < 0$ max we are looking for a max.

END pt.

$f(12) =$

(5)




$A(x) = x(120-2x)$

$A(x) = 120x - 2x^2$

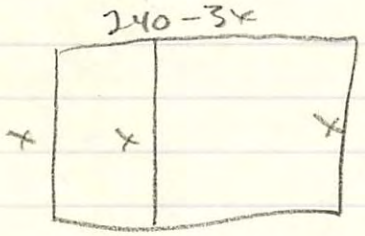
$A'(x) = 120 - 4x \quad x = 30$

$A''(x) = -4 \quad \therefore$ 30 is max.

1800

30 × 60

(6)




$A(x) = (240-3x)x$

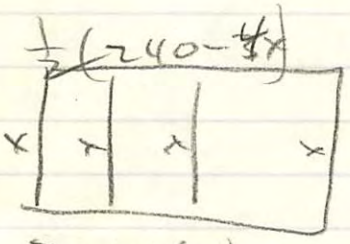
$A(x) = 240x - 3x^2$

$A'(x) = 240 - 6x \quad x = 40$

$A''(x) = -6 \quad \therefore$ 40 is max.

40 × 120

(7)




$A(x) = x(240-4x)$

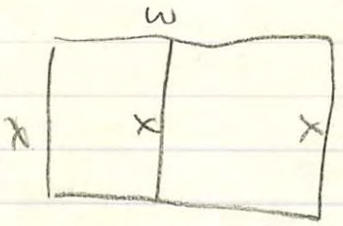
$A(x) = 240x - 4x^2$

$A'(x) = 240 - 8x \quad x = 30$

$A''(x) = -8 \quad \therefore$ 30 is max

30 × 60

(8)




$A = wx = 1350$ length is to be minimized

$L(x) = 3x + 2w$

$w = \frac{1350}{x}$

$L(x) = 3x + 2700x^{-1}$

$L'(x) = 3 - 2700x^{-2} = 0$

$L''(x) = 5400x^{-3}$

$L''(30) > 0 \quad \therefore$ min at 30.

$3x^2 = 2700$

$x^2 = 900$

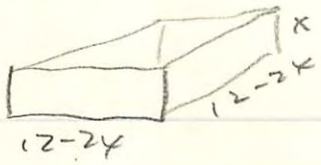
$x = 30$

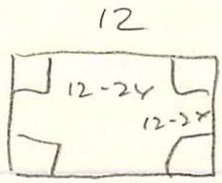
$\frac{1350}{30} \quad w = 45$

45 × 30

(9) $15\sqrt{3} \times 30\sqrt{3}$

(10)




$V(x) = (12-2x)^2 x$

$V(x) = 144x - 48x^2 + 4x^3$

$V'(x) = 144 - 96x + 12x^2 = 12(12 - 8x + x^2) = (x-6)(x-2)$

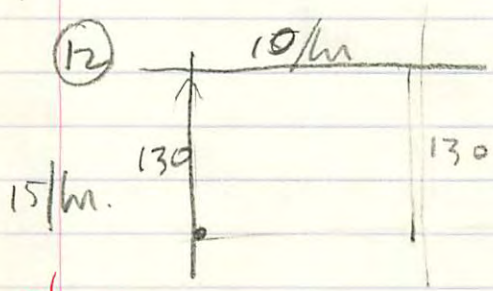
$V''(x) = 12 - 96 + 24x$

$V''(2) < 0$ $\therefore$ max volume when $x = 2$

reject $x = 6$

$2 \times 8 \times 8$

(12)




$d(t) = (10t)^2 + (130 - 15t)^2$

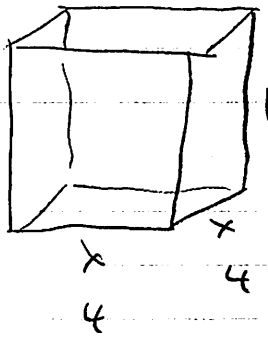
$100t^2 +$

6hrs

13 4hrs.

14 $3 \times 3 \times 4$

15




$\frac{64}{16} = 4$

$x^2 h = 64$

$h = \frac{64}{x^2} = 64x^{-2}$

$C(x) = 2x^2 + 4(64x^{-2})x$

$C = 2x^2 + 256x^{-1}$

$C'(x) = 4x - 256x^{-2} \overset{\text{set}}{=} 0$

$4x^3 = 256$

$x^4 = 64$

$x = 4$

Box is $4 \times 4 \times 4$

Bore cost \$32 each side, 4 sides.

64
32
\$96 total cost.

$2 \times 2 \times 16$ ?

Bore cost \$8 ea side, 4 sides

32
128
8
136 total cost.

Pg 164

(17) $x+2y=5$ $P(0,0)$

$(x,y)$ required.

$d^2 = x^2 + y^2$

$= (5-2y)^2 + y^2 = f$

(19) $L: 3x-2y+6=0$ $P(1,2)$ $\frac{3x+6}{2}=y$

$f(x)=d^2=(x-1)^2+(y-2)^2$

$f(x)=x^2-2x+1+\left(\frac{3x+6-4}{2}\right)^2$

$f(x)=x^2-2x+1+\left(\frac{3x+2}{2}\right)^2$

$f(x)=x^2-2x+1+\frac{1}{4}(9x^2+12x+4)$

$f(x)=\frac{13}{4}x^2+x+2$

$f'(x)=\frac{13}{2}x+1 \overset{\text{set}}{=} 0$

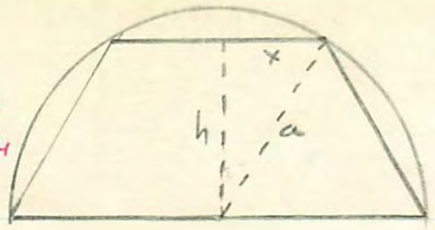
$x=-\frac{2}{13}$

$\frac{3\left(\frac{-2}{13}\right)+6}{2}=y$

$\frac{-\frac{6}{13}+\frac{78}{13}}{2}$

$\frac{36}{13}=y$

1st 5 show this with $a=5$ then go back and do it with $a$ as a variable




e.g. 3 50 with yellow chalk using

we want to find max area.

area of a trapezoid is $h\left(\frac{b_1+b_2}{2}\right)$

$h = \sqrt{a^2-x^2}$ by Pyth. thm.

thus $A(x) = \frac{1}{2}(2x+2a)\sqrt{a^2-x^2}$

$A(x) = (x+a)\sqrt{a^2-x^2} \quad 0 \le x \le a$

$$\frac{dA}{dx} = \frac{d(x+a)}{dx}\left(\sqrt{a^2-x^2}\right) + (x+a)\frac{d\left(\sqrt{a^2-x^2}\right)}{dx}$$

$$\frac{d\left(\sqrt{a^2-x^2}\right)}{dx} \Rightarrow \quad \text{let } u = a^2-x^2 \quad \text{let } v = u^{\frac{1}{2}}$$

$$\frac{du}{dx} = -2x \qquad \frac{dv}{du} = \frac{1}{2\sqrt{u}}$$

$$\frac{d\left(\sqrt{a^2-x^2}\right)}{dx} = \frac{(-2x)}{2\sqrt{a^2-x^2}} = -\frac{x}{\sqrt{a^2-x^2}}$$

$$\frac{dA}{dx} = \sqrt{a^2-x^2} - \frac{x(x+a)}{\sqrt{a^2-x^2}}$$

$$= \frac{a^2-x^2-x^2-ax}{\sqrt{a^2-x^2}}$$

$$= \frac{a^2-ax-2x^2}{\sqrt{a^2-x^2}}$$

$$= \frac{(a+x)(a-2x)}{\sqrt{a^2-x^2}} = \text{let } = 0$$

$x = -a$ or $x = +\frac{a}{2}$

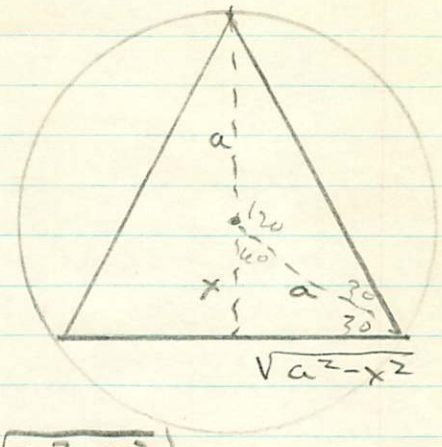
use f' test $\to \frac{a}{2}^- +$ $\to \frac{a}{2}^+ -$ $\to$ $\therefore$ max.

endpts yield min.

$A\left(\frac{a}{2}\right) = \frac{1}{2}(3a)\sqrt{a^2-\frac{a^2}{4}} \Rightarrow \frac{3a}{2}\sqrt{\frac{3a^2}{4}} = \frac{3a^2\sqrt{3}}{4}$

max area.

#17




$A = \frac{1}{2}(b\,h)$

$A = \frac{1}{2}(a+x)(2\sqrt{a^2-x^2})$

$A(x) = (a+x)(\sqrt{a^2-x^2})_{0 \le x < a}$

$A'(x) = \sqrt{a^2-x^2} + (a+x)\left(\frac{d\sqrt{a^2-x^2}}{dx}\right)$

Let $u = a^2-x^2$

Same as example 3 pg 181-182 at this pt.

triangle turns out to be equilateral.

Max area is $\frac{3\sqrt{3}}{4}a^2$

#18. Pg 184

$2\pi r h = S.$

$\frac{1}{2}h = \sqrt{a^2-r^2}$

$h = 2\sqrt{a^2-r^2}$

$(2\pi r)(\sqrt{a^2-r^2}) = S$

$u \quad v$

$v\,du + u\,dv = S'$

$du:$

$u = a^2 - r^2$

$\frac{du}{dr} = -2r$

$y = u^{\frac{1}{2}}$

$\frac{dy}{du} = \frac{1}{2u^{\frac{1}{2}}} = \frac{1}{2\sqrt{u}}$

$\frac{dy}{dR} = \frac{-2r}{2\sqrt{a^2-r^2}} = \frac{-r}{\sqrt{a^2-r^2}}$

$4\pi\sqrt{a^2-r^2} + 4\pi r\left(\frac{-r}{\sqrt{a^2-r^2}}\right) \overset{\text{set}}{=} 0$

$4\pi(a^2-r^2) - 4\pi r^2 = 0$

$a^2 - r^2 - r^2 = 0$

$-2r^2 = -a^2$

$r = \frac{a}{\sqrt{2}}$

$h = 2\sqrt{a^2 - \frac{a^2}{2}}$

$2\sqrt{\frac{a^2}{2}}$

$\frac{2a}{\sqrt{2}}\ \frac{\sqrt{2}}{\sqrt{2}}$

$\frac{2a\sqrt{2}}{2}$

$a\sqrt{2}$

$\frac{4\pi a}{\sqrt{2}}\sqrt{a^2 - \frac{a^2}{2}}$

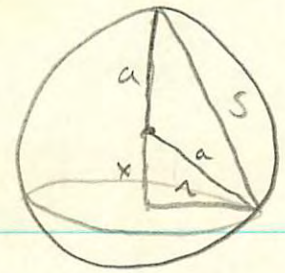
$\frac{4\pi a}{\sqrt{2}}\sqrt{\frac{2a^2}{2}}$

$\frac{4\pi a^2}{2} = 2\pi a^2$

$2\ \frac{u}{\sqrt{v}}$

$a\sqrt{2}$

19




$r = \sqrt{a^2 - x^2}$

$s = \sqrt{(x+a)^2 + (a^2 - x^2)}$

$x^2 + 2ax + a^2 + a^2 - x^2$

Given $A = \pi r s$

$A(x) = \pi \sqrt{a^2 - x^2} \sqrt{2a^2 + 2ax}$

$A'(x) = \frac{-\pi x \sqrt{2a^2+2ax}}{\sqrt{a^2-x^2}} + \frac{a\pi\sqrt{a^2-x^2}}{\sqrt{2a^2+2ax}} \stackrel{\text{let}}{=} 0$

$-\pi x(2a^2 + 2ax) + a\pi(a^2 - x^2) = 0$

$-2a^2x - 2ax^2 + a^3 - ax^2 = 0$

$3x^2 + 2ax - a^2 = 0$

$(3x - a)(x + a)$

$x = \frac{a}{3} \quad x = -a$ reject

$A'(x) = \frac{-a\pi(3x-a)(x+a)}{\sqrt{\quad}\sqrt{\quad}}$

$- \; 0 \; + \; 0 \; -$

$-a \qquad \frac{a}{3}$

max Area by f' test.

$x = \frac{a}{3} \therefore h = \frac{4a}{3}$ $\quad r = \sqrt{a^2 - \frac{a^2}{9}} = \frac{2a\sqrt{2}}{3}$

$f(x) = \sin 2x - 2\sin x \quad \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right]$

$f'(x) = 2\cos 2x - 2\cos x$

$f''(x) = -4\sin 2x + 2\sin x$

$-8\sin x\cos x + 2\sin x$

$-2\sin x(4\cos x - 1)$

$f' = 2(\cos 2x - \cos x)$

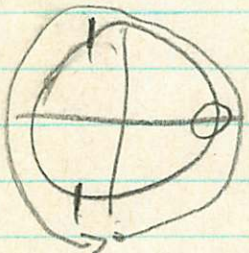
$2(\cos^2 x - \sin^2 x - \cos x)$

$2(2\cos^2 x - \cos x - 1) \overset{\text{set}}{=} 0$

$2(2\cos x + 1)(\cos x - 1) = 0$

$\cos x = -\frac{1}{2} \qquad \cos x = 1$

$x = \left\{0, \frac{2\pi}{3}, \frac{4\pi}{3}\right\}$



1st shows f'' test

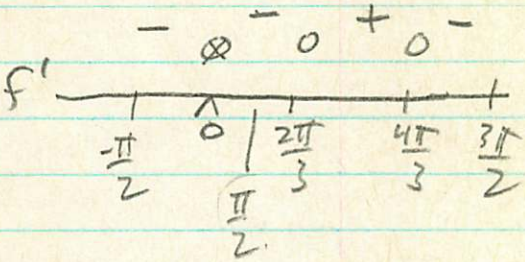
$f''(0) = -2(0)(4(1)-1) = 0$ test fails.

$f''\left(\frac{2\pi}{3}\right) = -2\left(\frac{\sqrt{3}}{2}\right)\left(4\left(-\frac{1}{2}\right)-1\right)$

$-\sqrt{3}(-3) > 0 \therefore \text{min.}$

$f''\left(\frac{4\pi}{3}\right) = -2\left(-\frac{\sqrt{3}}{2}\right)\left(4\left(-\frac{1}{2}\right)-1\right)$

$\sqrt{3}(-3) < 0 \therefore \text{max}$




$f'\left(\frac{\pi}{2}\right) = 2(\cos\pi) - 2\cos\frac{\pi}{2}$

$-2 - 0$

| x | f(x) | |
|---|---|---|
| $-\frac{\pi}{2}$ | 2 | |
| 0 | 0 | PI |
| $\frac{2\pi}{3}$ | $-\frac{3\sqrt{3}}{2}$ | |
| $\frac{4\pi}{3}$ | $-\frac{\sqrt{3}}{2}$ | |
| $\frac{3\pi}{2}$ | 2 | |

$f\left(-\frac{\pi}{2}\right) = 0 - 2(-1) = 2$

$f\left(\frac{2\pi}{3}\right) = \sin\frac{4\pi}{3} - 2\sin\frac{2\pi}{3}$

$-\frac{\sqrt{3}}{2} - 2\left(\frac{\sqrt{3}}{2}\right) = -\frac{3\sqrt{3}}{2}$

$f\left(\frac{4\pi}{3}\right) = \sin\frac{8\pi}{3} - 2\sin\frac{4\pi}{3}$

$\frac{\sqrt{3}}{2} - 2\left(\frac{\sqrt{3}}{2}\right) = -\frac{\sqrt{3}}{2}$

$f\left(\frac{3\pi}{2}\right) = \sin 3\pi - 2(-1) = 2$

# Implicit differentiation.

Implicit differentiation is used when even it may be difficult or impossible to attain only one variable in a fctn we wish to manipulate for eg.
Remember we put our fctn in one variable the found $f'(x)$ set $=0$ etc.
the text has an elementary eg.

Given $x^2+y^2=25$ find $\frac{dy}{dx}$ at $(3,4)$

the usual procedure is

Solve for $y$ $\quad y^2 = 25-x^2$

$y = \pm\sqrt{25-x^2}$

Notice a problem $\pm$. why?
because our given is not a fctn.
So, since we are interested in 1st Quad so we take $y = +\sqrt{25-x^2}$
& solve for $y'$ using chain rule $y' = \frac{-x}{\sqrt{25-x^2}}$ etc.
Now substitute (3,4) to find slope at that point.

using implicit differentiation

$$\frac{d(x^2+y^2)}{dx} = \frac{d(25)}{dx}$$

$$\frac{d(x^2)}{dx} + \frac{d(y^2)}{dx} = 0$$

$$2x + 2y\frac{dy}{dx} = 0$$

$$2y\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = -\frac{x}{y}$$

the pt in question was $(3,4)$ $\therefore$ answer is $-\frac{3}{4}$.

Consider using chain rule

$$z = \sqrt{x^2+a^2}$$

Let $y = x^2+a^2 \quad z = y^{\frac{1}{2}}$

$$\frac{dy}{dx} = 2x \qquad \frac{dz}{dy} = \frac{1}{2\sqrt{y}} = \frac{1}{2\sqrt{x^2+a^2}}$$

$$\therefore \frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx}$$

$$\text{So } \frac{dz}{dx} = \frac{x}{\sqrt{x^2+a^2}}$$

But what if you didn't know that $y = x^2+a^2$? ~~[illegible]~~,
the derivative of $y$ wrt $x$ is still $\frac{dy}{dx}$ whether we know what $y$ is or not

$$\text{So } z = y^{\frac{1}{2}}$$

$$\frac{dz}{dx} = \frac{1}{2} y^{-\frac{1}{2}} \cdot \frac{dy}{dx}$$

# Implicit Diffs

$x^2 + y^2 = 25$ $\quad (-3, 4)$

$y = \pm\sqrt{25-x^2} = +\sqrt{25-x^2}$ top $\frac{1}{2}$

$y' = \frac{1}{2}(25-x^2)^{-\frac{1}{2}}(-2x)$

$= \left.\frac{-x}{\sqrt{25-x^2}}\right| = \frac{3}{4}$

$x=-3$

$y - y_1 = m(x-x_1)$

$y - 4 = \frac{3}{4}(x+3)$

$y = \frac{3}{4}x + \frac{9}{4} + \frac{16}{4}$

$y = \frac{3}{4}x + \frac{25}{4}$

This method was called explicit

Now Done Implicitly.

$x^2 + y^2 = 25$

$2x + 2y\frac{dy}{dx} = 0$

and the rest is the same.

$\frac{dy}{dx} = -\frac{x}{y}\Big| = \frac{3}{4}$

$4 + 2 + 1$ $\qquad (-3, 4)$

Eg 3 $\quad x^2 + x^u y^v + y^5 = 7 \quad (2, 1)$

$2x + y + x\frac{dy}{dx} + 5y^4\frac{dy}{dx} = 0$

$(x + 5y^4)\frac{dy}{dx} = -(2x+y)$

$\frac{dy}{dx} = \left.\frac{-(2x+y)}{x+5y^4}\right| = \frac{-5}{7}$

$(2, 1)$

# Implicit Diff.

Looking at a more complicated problem

$x^2 + xy + y^5 = 3$ find slope at $(1,1)$

$$\frac{d(x^2)}{dx} + \frac{d(xy)}{dx} + \frac{d(y^5)}{dx} = \frac{d(3)}{dx}$$

$$2x + \left(y + x\frac{dy}{dx}\right) + 5y^4\frac{dy}{dx} = 0$$

$$(x + 5y^4)\frac{dy}{dx} = -(2x + y)$$

$$\frac{dy}{dx} = -\frac{2x+y}{x+5y^4}$$

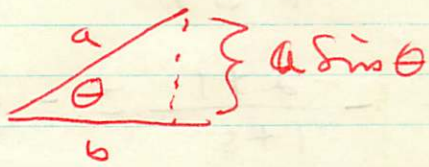
$$\left.\frac{-2+y}{x+5y^4}\right]_{(1,1)} = -\frac{2+1}{1+5} = -\frac{3}{6} = -\frac{1}{2}$$

4.

2 sides of a triangle and their included ~~side~~ angle are changing w.r.t. time. The angle increases at the rate of 1 radians per second; one side increases at the rate of 3 ft/sec and the other side decreases at the rate of 2 ft/sec. Find the rate at which the area is changing when $\theta = \frac{\pi}{4}$

$a = 4$

$b = 5$. Is the area increasing or decreasing?




(1) Picture

(2) List Given

$\frac{d\theta}{dt} = 1$

$\frac{da}{dt} = 3$

$\frac{db}{dt} = -2$ ↑

(3) List Question

$\frac{dA}{dt}\Big|$

$\theta = \frac{\pi}{4}$

$a = 4$

$b = 5$

(4) Get relationship between A and any or all of $\theta, a, b$

$A = \left(\frac{1}{2}ab\right)\sin\theta$

$\frac{dA}{dt} = \sin\theta\left[\frac{1}{2}b\frac{da}{dt} + \frac{1}{2}a\frac{db}{dt}\right] + \frac{1}{2}ab\cos\theta\frac{d\theta}{dt}$

$= \sin\frac{\pi}{4}\left[\frac{1}{2}\cdot 5\cdot 3 + \frac{1}{2}\cdot 4(-2)\right] + \frac{1}{2}\cdot 4\cdot 5\cos\frac{\pi}{4}\cdot 1$

$= \frac{\sqrt{2}}{2}\left[\frac{15}{2} - \frac{8}{2}\right] + 10\frac{\sqrt{2}}{2}$

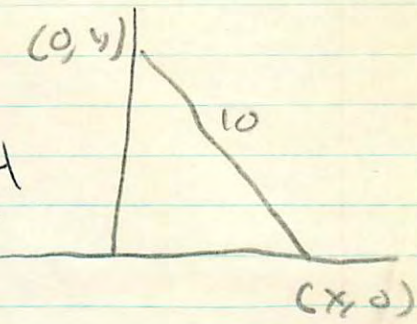
$= \left(\frac{7}{2} + \frac{20}{2}\right)\frac{\sqrt{2}}{2}$

$= \left(\frac{27}{2}\right)\frac{\sqrt{2}}{2}$

$= \frac{27\sqrt{2}}{4}$ ✓ Sentence Answer! ✓

Increasing because $\frac{dA}{dt} > 0$ ✓

eg 3 pg 187




$\frac{dx}{dt} = 3$ given

find $\frac{dy}{dt}$

∴ need any valid relationship between x & y.

$x^2 + y^2 = 10^2$

$\frac{d(x^2)}{dt} + \frac{d(y^2)}{dt} = \frac{d(10^2)}{dt}$

$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$ but $\frac{dx}{dt} = 3$

$6x + 2y\frac{dy}{dt} = 0$

$2y\frac{dy}{dt} = -6x$

$\frac{dy}{dt} = -\frac{3x}{y}$

Question was what is rate when golden is 8' from ground. If $y = 8$ then $x = 6$ from Pythag. Thm.

thus $\frac{dy}{dt} = -\frac{3(6)}{8} = -\frac{9}{4}$

it moving <u>down</u> at a rate of $\frac{9}{4}$ ft/sec

Related rates (applications of chain rule).

Eg 1. Given rate of change in volume, in terms of time

$$\frac{dV}{dt} = 100.$$

we wish to know what rate the radius is changing in terms of time.

$V = \frac{4}{3}\pi r^3$ Volume of sphere

$\frac{dV}{dr} = 4\pi r^2$ Volume changing in terms of radius.

we need $\frac{dr}{dt}$

$$\frac{dr}{dt} = \frac{dV}{dt} \cdot \frac{dr}{dV}$$

$$\frac{dr}{dt} = 100 \cdot \frac{1}{4\pi r^2}$$ Section a $r = 10$

$$\frac{dr}{dt} = \frac{25}{\pi r^2}$$

$$\frac{25}{\pi 100} = \frac{1}{4\pi}$$

Section b $r = 20$

$$\frac{dr}{dt} = \frac{25}{\pi 400} = \frac{1}{16\pi}$$

we have been using $f'(x)$

it is also written $\frac{dy}{dx}$

"The derivative of Y WRT X."

"The ~~da~~ rate of change of Y per change in X.

Consider square side E.

$A = E^2$ So rate of change of A per change in E is

$\frac{dA}{dE}$ is $2E$. sq inches.

~~How about when E is 4~~

Consider E getting larger (expanding as if heated) when E is 4 units the area of the square is changing at the rate of 8 sq inches

Orales pg 187

① $2s^2 = D^2$

$s\sqrt{2} = D$

$\sqrt{2}\frac{ds}{dt} = \frac{dD}{dt}$

$3\sqrt{2} = \frac{dD}{dt}$

② $P = 4s$

$\frac{dP}{dt} = 4\frac{ds}{dt}$

$\frac{dP}{dt} = 12$

③ $A = s^2$

$\frac{dA}{dt} = 2s\frac{ds}{dt}$

$= 2 \cdot 5 \cdot 3$

$= 30\ ft^2/sec$

④ $d = 2r$

$\frac{dd}{dt} = 2\frac{dr}{dt}$

$6 = 2\frac{dr}{dt}$

$3 = \frac{dr}{dt}$

⑤ $\pi D = C$

$\pi\frac{dd}{dt} = \frac{dC}{dt}$

$6\pi = \frac{dC}{dt}$

⑥ $A = \pi\left(\frac{d}{2}\right)^2$

$A = \frac{\pi d^2}{4}$

$\frac{dA}{dt} = \frac{\pi}{2} d \frac{dd}{dt}$

$\frac{dA}{dt} = \frac{\pi}{2} \cdot 2 \cdot 6$

$\frac{dA}{dt} = 6\pi$

① $x^2+y^2+4x-y=0$ , $(0,0)$

$$2x + 2y\frac{dy}{dx} + 4 - \frac{dy}{dx} = 0$$

$$(2y-1)\frac{dy}{dx} = -(2x+4)$$

$$\frac{dy}{dx} = -\frac{2(x+2)}{2y-1}\Bigg]_{0,0} = \frac{-4}{-1} = 4$$

③ $-x^2+y^3+5y^2-2=0$ $(4,-3)$

$$-2x + 3y^2\frac{dy}{dx} + 10y\frac{dy}{dx} = 0$$

$$(3y^2+10y)\left(\frac{dy}{dx}\right) = 2x$$

$$\frac{dy}{dx} = \frac{2x}{3y^2+10y}\Bigg]_{4,-3} = \frac{8}{27-30} = -\frac{8}{3}$$

⑤ $x^4+xy+y^4=19$ $(1,2)$

$$4x^3 + y + x\frac{dy}{dx} + 4y^3\frac{dy}{dx} = 0$$

$$x+4y^3\frac{dy}{dx} = -4x^3+y$$

$$\frac{dy}{dx} = -\frac{4x^3+y}{x+4y^3}\Bigg]_{1,2} = -\frac{6}{33} = \frac{-2}{11}$$

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#6 $x^3 - y^2 + 2x^2y^3 = 2 \quad (1,1)$

#7 $V = s^3$

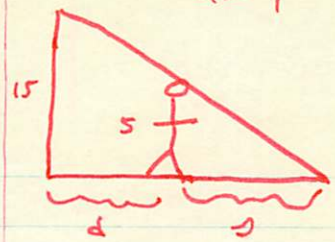
$\frac{dV}{dt} = 6$ $\frac{dV}{ds} = 3s^2$

when $V = 1000$
$s = 10$

$\frac{ds}{dt} = \frac{dV}{dt} \cdot \frac{ds}{dV}$

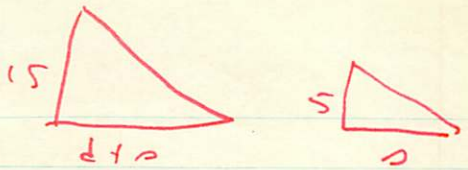
$= 6 \cdot \frac{1}{3s^2} = \frac{2}{10^2} = \frac{1}{100} = \frac{1}{50}$ cm/hr.

#9




$\frac{ds}{dt} = ?$

$\frac{dd}{dt} = 6$




$\frac{dA}{dt} = \frac{ds}{dt} \cdot \frac{dd}{ds}$

$\frac{15}{d+s} = \frac{5}{s}$

$15s = 5d + 5s$

$10s = 5d$

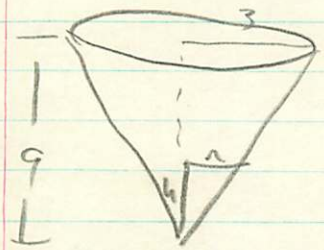
$2s = d$

$2 = \frac{dd}{ds}$

$6 = \frac{ds}{dt} \cdot 2$

$3 = \frac{ds}{dt}$

#11




$\frac{dh}{dt} = ?\Big]_{h=3}$

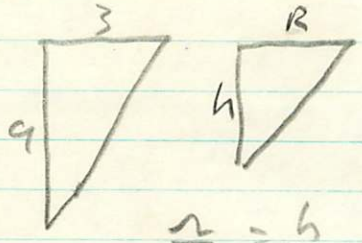
$\frac{dv}{dt} = 2$

$V = \frac{1}{3}\pi r^2 h$

$\frac{dv}{dt} = \frac{dh}{dt}\,\frac{dv}{dh}$

$V = \frac{1}{3}\pi r^2 h$

$\frac{dv}{dh} = \frac{2\pi r h}{3}\frac{dr}{dh} + \frac{\pi r^2}{3}$




$\frac{r}{3} = \frac{h}{9}$

$3r = h$

$3\frac{dr}{dh} = 1$

$\frac{dr}{dh} = \frac{1}{3}$

$\frac{2\pi r h}{9} + \frac{\pi r^2}{3}\Big]_{\substack{h=3 \\ r=1}} = \pi = \frac{dv}{dh}$

$2 = \frac{dh}{dt}\pi$

$\frac{2}{\pi} = \frac{dh}{dt}$

$V = \frac{1}{3}\pi r^2 h$

12. $\frac{dV}{dt} = 10$ $\left.\frac{dh}{dt}\right|_{h=5}$

$h = \frac{1}{4}(2r)$

$h = \frac{r}{2}$

$2h = r$ $\therefore$ $V = \frac{1}{3}\pi(4h^2)h$

$V = \frac{4}{3}\pi h^3$

$\frac{dV}{dt} = 4\pi h^2 \frac{dh}{dt}$

$10 = 4\pi(5)^2 \frac{dh}{dt}$

$\frac{10}{100\pi} = \frac{dh}{dt}$

$\frac{1}{10\pi} = \frac{dh}{dt}$.

I

a) $(f(g(x)))'\Big|_{x=0}$

$f'(g(0)) \cdot g'(0)$

$f'(2) \cdot 1$

$1 \cdot 1$

$1$

b) $(f(g))'\Big|_{x=2}$

$f'(g(2)) g'(2)$

$f'(2) \cdot 1$

$1 \cdot 1$

$1$

c) $(h \circ f)'\Big|_{0}$

$h'(f(0)) f'(0)$

$h'(1) \quad f'(2)$

$2 \cdot 1$

$2$

d) $(g(f))'\Big|_{x=1}$

$g'(f(1)) f'(1)$

$g'(0) \quad 1$

$1 \quad 1$

$1$

e) $(f(g))'\Big|_{x=1}$

$f'(g(1)) g'(1)$

$f'(1) \quad 0$

$0 \cdot 1$

$0$

f) $(g(f))'\Big|_{x=1}$

$g'(f(1)) f'(1)$

$g'(0) \quad 1$

$1 \quad 1$

$1$

g) $(g(f(h)))'\Big|_{2}$

$g'(f(h)) \cdot f'(h) \cdot h'$

$g'(f(0))$

$g'(1)$

$0 \cdot 0$

$0$

h) $(f(h(g)))'\Big|_{x=1}$

$f'(h(g)) h'(g) g'$

$f'(2) \cdot 1 \cdot 0$

$0 \cdot 1 \cdot 1$

$0$

i) $g(h(f))'\Big|_{x=0}$

$g'(h(f)) h'(f) f'$

$g'(2) \cdot 1 \cdot 2$

$1 \quad 2$

$2$

k) $(f f f(x))'\Big|_{x=0}$

$f' \cdot f'(f) f'(f(f))$

$1 \quad 1 \quad 2$

$2$

1, 3, 4

I A) $y = (3x^2+1)^{\frac{1}{3}}$

$u = 3x^2+1 \qquad y = u^{\frac{1}{3}}$

$\frac{du}{dx} = 6x \qquad \frac{dy}{du} = \frac{1}{3}u^{-\frac{2}{3}}$

$\frac{dy}{dx} = \frac{du}{dx} \cdot \frac{dy}{du}$

$= \frac{6x}{3(3x^2+1)^{\frac{2}{3}}}$

$= \frac{2x}{(3x^2+1)^{\frac{2}{3}}}$

B) $y = \frac{x^3+2x-1}{x^2+1}$

$u = x^3+2x-1 \qquad v = x^2+1$

$du = 3x^2+2 \qquad dv = 2x$

$\frac{v\,du - u\,dv}{v^2}$

$\frac{(x^2+1)(3x^2+2) - (x^3+2x-1)2x}{(x^2+1)^2}$

$\frac{3x^4+5x^2+2 - 2x^4-4x^2+2x}{(x^2+1)^2}$

$\frac{x^4+x^2+2x+2}{(x^2+1)^2} = \frac{x^2(x^2+1)+2(x+1)}{}$

C) $xy^2+x^2y=1$

$2xy\frac{dy}{dx} + y^2 + x^2\frac{dy}{dx} + 2xy = 0$

$(2xy+x^2)\frac{dy}{dx} = -(y^2+2xy)$

$\frac{dy}{dx} = \frac{-(y^2+2xy)}{x^2+2xy}$

D) $y = (4x+1)(x^3-2x+3)^4$

$u = 4x+1 \qquad v = (x^3-2x+3)^4$

$du = 4 \qquad dv = 4(3x^2-2)(x^3-2x+3)$

$\frac{dy}{dx} = 4(x^3-2x+3)^4 + 4(4x+1)(3x^2-2)(x^3-2x+3)$

$z = (x^3-2x+3)^4$

$u = x^3-2x+3$

$\frac{du}{dx} = 3x^2-2$

$z = u^4$

$\frac{dz}{du} = 4u^3 =$

$\frac{dz}{du} \cdot \frac{du}{dx} = 4(x^3-2x+3)^3(3x^2-2)$

II $x = t + \frac{1}{t}$ $\quad y = t - \frac{1}{t}$

$\frac{d^2y}{dx^2}$ -

$\frac{dx}{dt} = 1 - t^{-2}$ $\quad \frac{dy}{dt} = 1 + t^{-2}$

$(t^2-1)^2 t \frac{dt}{dx}$ -

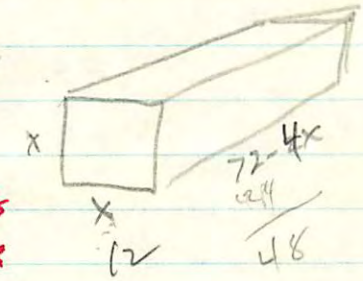
$\frac{d^2x}{dt^2} = 2t^{-3}$ $\quad \frac{d^2y}{dt} = -2t^{-3}$

$\frac{dy}{dx} = \frac{1 + \frac{1}{t^2}}{1 - \frac{1}{t^2}} = \frac{t^2+1}{t^2-1}$ u / v

changed

NO! $\frac{d^2y}{dx^2} = \frac{-2t^{-3}}{2t^{-3}} = -1$

wrong III




$V = x^2(72 - 4x)$

$V(x) = 72x^2 - 4x^3$

$V'(x) = 144x - 12x^2 \overset{\text{set}}{=} 0$

$12x(12 - x) = 0$

$x = 0 \quad x = 12$

$V''(x) = 144 - 24x$

$12 \cdot 12 - 12 \cdot 24$

$12(12 - 24) < 0 \therefore$ MAX

Volume at $x = 24$

dimensions

~~24 x 24 x 24~~

12 x 12 x 24

IV $f(x) = 4x^3 - 12x + 17 \quad [0, 2]$

$f'(x) = 12x^2 - 12x = 12(x^2 - 1)$

$f''(x) = 24x - 12 = 12(2x - 1)$

$x = 1$ f at max relative,

$x = -1$ f at min absolute,

$x = 0$ rel max (because after it f is decreasing.)

crit values $\pm 1$

| | | | | |
|---|---|---|---|---|
| f → | 4 | 0 | -12 | 7 |
| 0 | | | | 7 |
| 2 | 4 | 8 | 4 | 15 |
| 1 | 4 | 4 | -8 | -1 |
| -1 | 4 | -4 | -8 | 15 |

I $y = x^3 - 6x^2 + 3x + 5$

$y' = 3x^2 - 12x + 3$ ← rate of change of fctn,

$y'' = 6x - 12$ ← rate of change of slope

$y' \overset{\text{Set}}{=} y''$

$3x^2 - 12x + 3 = 6x - 12$

$3x^2 - 18x + 15 = 0$

$x^2 - 6x + 5 = 0$

$(x-1)(x-5) = 0$

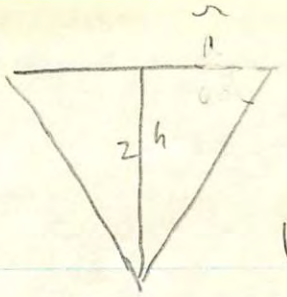
$x \in \{1, 5\}$

| | | | | |
|---|---|---|---|---|
| $y \to$ | 1 | -6 | 3 | 5 |
| 1 | 1 | -5 | -2 | 3 |
| 5 | 1 | -1 | -2 | -5 |

(1, 3) (5, -5) are the pts where rate of change of ordinate is = to rate of change of slope,

Importance of this pblm: understanding that rate of change of an item is the item's 1st Derivative!

VI




$\frac{dV}{dt} = 2$ $\qquad \frac{dh}{dt} = ?$

$V = \frac{1}{3}\pi r^2 h$

$2r = h$

Since $r = 1$ when $h = 2$ so $r = \frac{h}{2}$

$$V(h) = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h = \frac{1}{12}\pi h^3$$

$$\frac{dV}{dt} = \frac{1}{4}\pi h^2 \frac{dh}{dt}$$

$$2 = \frac{1}{4}\pi h^2 \frac{dh}{dt}$$

$$\frac{8}{\pi h^2} = \frac{dh}{dt}\Bigg]_{h=?}$$

we need to determine h when cone is $\frac{1}{8}$ filled.

when full $h = 2$, $r = 1$ so $V = \frac{1}{3}\pi \cdot 1^2 \cdot 2 = \frac{2\pi}{3} = V$

$\frac{1}{8}V = \frac{1}{8}\cdot\frac{2\pi}{3} = \frac{\pi}{12} = V$, we need to find h when $V = \frac{\pi}{12}$ so: $\frac{\pi}{12} = \frac{1}{3}\pi r^2 h = \frac{\pi}{12} h^3$

$\therefore h^3 = 1$ so $h = 1$

$$\frac{dh}{dt} = \frac{8}{\pi h^3}\Bigg]_{h=1} = \frac{8}{\pi}$$ ft/min is the rate the height is changing when the vessel is $\frac{1}{8}$ filled.

HONORS ANALYSIS CHAPTER 4 APPLICATIONS

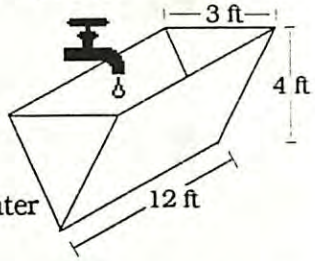
# APPLYING THE DERIVATIVE

**I** Find $\frac{dy}{dx}$ for each of the following: A) $y^3 = 3x^2 + 1$ B) $y = \frac{x^3 + 2x - 1}{x^2}$ C) $xy^2 + x^2y = 1$

D) $y = (4x+1)(x^3 - 2x + 3)^4$ E) $y = \frac{1}{2}(\sin x \cos x + x)$ F) $y = \sin x - x\cos x$

G) $2y = x - \sin x \cos x$

**II** The length of the water trough is 12 feet. It was made by bending some metal in the shape of a V and then the ends were soldered to the trough to make it water tight. The cross section of the trough ( i.e. each end of the trough) is an isosceles triangle with measurements as shown in the picture. The water faucet delivers water at 3 cubic feet per minute.




A) Assume that the trough has already been filled to a depth of 2 feet ( as measured from the apex of the triangle to the surface of the water.) You watch reads 2:20 in the afternoon. What time will it be when the trough is completely full?

B) How fast was the depth changing at the instant that the trough became full?

**III** A package to be sent in the mail can have a combined girth ( distance around) plus length of at most 72 inches. What is the volume of the largest box ( maximum volume ), with a square base, which can be sent?

**IV** Find the extrema of $F(x) = 4x^3 - 12x + 7$ for $x \in [0, 2]$

**V** Along the graph of the equation $y = x^3 - 6x^2 + 3x + 5$ both the ordinate "y" and the slope ( of the tangent) "m" change, but generally at different rates. Find the point (or points), if any, where the ordinate and the slope are momentarily changing at the *SAME* rate.

**VI** Water flows at the rate of 2 cubic feet per minute into a vessel in the form of an inverted right circular cone of altitude 2 feet and base radius 1 foot. At what rate is the surface of the water rising when the vessel is one-~~third~~ eighth full?

**VII** Since 1980 the enrollment at Vacaville High School is approximated by the formula given as "V". The student population of Wood High School is given by the formula "W". What will be the maximum enrollments for each school? When will it occur? ( t is the the number of years since 1980) What is the current enrollments?

$V(t) = 7000 - (t^2 - 28t + 196)$ $W(t) = 10000 + 50000(t + 10)^{-1}$

**VIII** One leg (AB) of right triangle ABC increases at the rate of 2 in/sec while the other leg decreases at the rate of 3 in/sec. Determine how fast the hypotenuse is changing AND state whether it is getting longer or shorter when leg AB = 6 and leg AC = 8. What about the area of the triangle at the same time?

**IX** Find the altitude of the isosceles triangle with largest area which can be inscribed inside a circle with radius 5.

**X** A man is flying a kite which is a constant 50 feet above the ground. A horizontal wind is blowing the kite away from the man. When 130 feet of string has been let out, the kite is pulling the string out at a rate of 20 ft/sec. What is the wind velocity?

**XI** We are considering investing a large sum of money. Our broker brought us the figures for two companies. Acme Unlimited and Baker Industries. The net assets are in units of 1 billion dollars.

$$\text{Acme: } A(t) = 3t(t+1)^{-1} \quad \text{Baker: } B(t) = t^2(t+1)^{-1} + 1$$

A) Find the assets for each company one year from today ( ie. when t = 1 ).

B) Which company should we invest with? Why?

C) Determine the growth rate ( ie. the rate of change in the assets = growth rate) for each company.

D) Evaluate each of these at t = 1.

E) Are you now able to decide which company is the best choice (from the invester's point of view)? Why?

F) Let's consider how the "growth rate" is changing. Since the first derivative of assets is growth rate, it seems logical that the first derivative of growth rate will tell us how active the growth rate is. So, find the second derivative of each function.

G) Evaluate both at t = 1.

H) When the second derivative is positive, we say the business is growing ( ie. the growth rate is accelerating). ( We are holding the gas pedal of the car completely down.) When the second derivative is negative we say the growth rate is decelerating. ( To continue the analogy, this is like removing your foot from the gas pedal. The car is still traveling but is slowing at the same time.) We have seen that the asets of both companies will be the same 1 year from now, and their growth rates are the same. However, we can see that one of the businesses will increase as time marches on while the other ...

I) Should we choose a company with a currently high growth rate that is decreasing slowly, or one with a smaller growth rate that is increasing slowly? Why?

J) Speculate on the idea that the business with is decreasing has currently a higher return than the business which is increasing -- that is -- which would be the best choice? Why?

**XII** The ( x , y ) coordinate of a point *P* is given by: $x = t + \frac{1}{t}$ ; $y = t - \frac{1}{t}$

A) Find $\frac{dy}{dx}$ ( HInt: you are using the definition of *Chain Rule* )

B) Find $\frac{d^2y}{dx^2}$ ( This is the second derivative of y with respect to x. Answer is NOT – 1 )

C) Determine the equation involving x and y only. ( ie. get rid of the variable t )

D) What conic section is this? Draw its graph -- ( Ask someone in Alg II -- They can help. )

**XIII** The sum of two non-negative numbers is 1.

A) The sum of the square of the first and the cube of the second is a maximum. Find both numbers.

B) The sum of the square of the first and the cube of the second is a minimum. Find both numbers.

**XIV** Given rectangle *ABCD* has 10 sq ft of area. Point *E* is the mid-point of side *BC*. Determine the dimensions of *ABCD* such that the distance from *A* to *E* is a minimum.

**XV** A rectangle has two of its verticies on the x-axis and the other two above the x-axis and on the graph of the parabola: $y = 16 - x^2$ . Of all such possible rectangles, find the dimensions of the one with maximum area. Of all such possible rectangles, find the dimensions of the one with maximum perimeter.

1959
21

## VII

a) $e(\tau) = 7000 - (\tau^2 - 28\tau + 196)$

$e'(\tau) = -2\tau + 28 \overset{\text{Set}}{=} 0$

$\tau = 14$

max enrollment in 1972 which will be 7000 students.

$e(\tau) = 7000 - (\tau - 14)^2$ ENding − Beginning + 1 = # of Yrs

$= 7000 - (8)^2$

$= 7000 - 49$

$= 6951$ students today.

b) $e(\tau) = 10{,}000 + \frac{50{,}000}{\tau + 10}$ $u = 50{,}000$ $v = \tau + 10$ $du = 0$ $dv = 1$

$e'(\tau) = \frac{-(50{,}000)}{(\tau+10)^2}$ There is no achievable max. Voca U had its max when $\tau = 0$ at 15,000 when it opened.

today Voca U has 11,563 students. ($\tau = 22$)

## VIII

$a_1(\tau) = \frac{3\tau}{\tau+1}$ $a_2(\tau) = \frac{\tau^2}{\tau+1} + 1$

① $a_1(\tau)\Big]_{\tau=1} = \frac{3 \cdot 1}{1+1} = \frac{3}{2}$ $a_2(\tau)\Big]_{\tau=1} = \frac{1}{2} + 1 = \frac{3}{2}$, $1\frac{1}{2}$ Billion $

② No choice can be made

③ $u = 3\tau$ $v = \tau + 1$ $du = 3$ $dv = 1$

$\frac{3\tau + 3 - 3\tau}{(\tau+1)^2} = \frac{3}{(\tau+1)^2} = a_1'(\tau)$

$u = \tau^2$ $v = \tau + 1$ $du = 2\tau$ $dv = 1$

$\frac{2\tau^2 + 2\tau - \tau^2}{(\tau+1)^2} = \frac{\tau^2 + 2\tau}{(\tau+1)^2} = a_2'(\tau)$

④ $\frac{3}{(\tau+1)^2}\Big]_{\tau=1} = \frac{3}{(2)^2} = \frac{3}{4}$

$\frac{\tau^2 + 2\tau}{(\tau+1)^2}\Big]_{\tau=1} = \frac{1+2}{(2)^2} = \frac{3}{4}$

Both are growing at 3/4 Billion $/yr at $\tau = 1$,

⑤ No. Both are the same.

VIII cont.

(6) (a₁)

$a_1'(\tau) = \frac{3}{(\tau+1)^2}$

$u = 3 \quad v = (\tau+1)^2$

$du = 0 \quad dv = 2\tau + 2$

$u = \tau + 1 \quad z = u^2$

$\frac{du}{d\tau} = 1 \quad \frac{dz}{du} = 2u$

$\frac{da}{d\tau} = 1 \cdot 2u = 2(\tau+1) = 2\tau + 2$

$a_1''(\tau) = \frac{-6\tau - 6}{(\tau+1)^4} = \frac{-6(\tau+1)}{(\tau+1)^4} = \frac{-6}{(\tau+1)^3}$

(a₂) $a_2'(\tau) = \frac{\tau^2 + 2\tau}{(\tau+1)^2}$

$u = \tau^2 + 2\tau \quad v = (\tau+1)^2$

$du = 2\tau + 2 \quad dv = 2\tau + 2$

$\frac{(\tau+1)^2(2\tau+2) - (\tau^2+2\tau)(2\tau+2)}{(\tau+1)^4} = \frac{2(\tau+1)^3 - 2\tau(\tau+1)(\tau+2)}{(\tau+1)^4} =$

$2\tau^2 + 4\tau + 2 - 2\tau^2 - 4\tau$

$\frac{2(\tau+1)^2 - 2\tau(\tau+2)}{(\tau+1)^3} = a_2''(\tau) = \frac{2}{(\tau+1)^3}$

(7)

$a_1''(\tau) = \left.\frac{-6}{(\tau+1)^3}\right]_{\tau=1} = \frac{-6}{2^3} = -\frac{3}{4}$ ← $a_1''$

$a_2''(\tau) = \left.\frac{2(\tau+1)^2 - 2\tau(\tau+2)}{(\tau+1)^3}\right]_{\tau=1}$

$\frac{2(2)^2 - 2(3)}{2^3} = \frac{8-6}{8} = \frac{1}{4}$ ← $a_2''(1)$

(8) decreases as time moves along.

(9) Depends on length of investment.

ANALYSIS ch 4 — "A Very Serious Ditto" — Mumaugh

Respond using complete sentences on the following (where applicable)

① General recognition method is How one is to identify the type of problem.
② Overview of method.
③ Step by step explanation similar to two column proofs in geometry. Same intuitive sense in why a step was used rather than what was actually done.
~~④ How method up at beginning of lesson to [illegible] comment.~~
~~⑤ The greatest [illegible] completion.~~

#1 A) Pg 164 Pblm 15.
B) IN PLACE ← { IN PLACE of Thm #1 use: $\frac{d(uv)}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$

Pg 178 19-22

#2 A) Pg 177 Pblm 14.
B) using $\frac{d\left(\frac{u}{v}\right)}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$, find $\frac{dz}{dx}$

where $z = \frac{PQ}{RS}$. P, Q, R, S are fctns of x.

C) write out question & answers to odds Pg 183. Explain your method on EACH Pblm.
D) Pg 183 pblm 5 (2 methods)
pblm 16

#3

↓

#3

A) Pg 164 Prblm 20

B) Pg 164 prblm 18

C) Given $y = \sqrt{\frac{3x^2-4x-4}{x-2}}$ $\forall x \in D_y \cap [-\frac{1}{2}, 3]$

Locate & identify all extrema. Sketch. Verify convexity. Justify!!

D) Given $3x^2+4y^2=12$

Same instructions as C. Also Determining the coordinates where the tangent line to the curve is $\parallel$ to: $2y-x=2$.

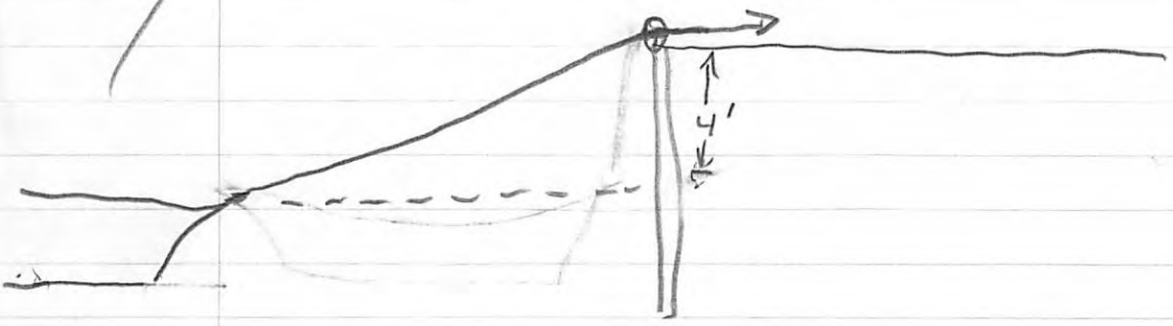
E) A boat is pulled into a dock by means of a rope w/ one end attached to the bow of the boat, the other end passing through a ring attached to the dock at apt 4' higher than the bow of the dock. If the rope is pulled in at a rate of 2 ft/sec, how fast is the boat approaching the dock when 10 ft of rope are out?

$(1, \frac{3}{2})$

$(1, -\frac{3}{2})$

$\frac{10\sqrt{21}}{21}$

Do not include these



Ix $x = t + \frac{1}{t}$ $\quad y = t - \frac{1}{t}$

A) $\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$ $\qquad \frac{dx}{dt} = 1 - \frac{1}{t^2}$

$\frac{dy}{dt} = 1 + \frac{1}{t^2}$

$\frac{dy}{dx} = \frac{t^2+1}{t^2} \cdot \frac{t^2}{t^2-1}$

$\frac{t^2+1}{t^2-1}$ u

v

B) $\frac{d\left(\frac{dy}{dx}\right)}{dx} = \frac{(t^2-1)2t\frac{dt}{dx} - (t^2+1)2t\frac{dt}{dx}}{(t^2-1)^2}$

$= \frac{(t^2-1-t^2-1)2t\frac{dt}{dx}}{(t^2-1)^2}$

$= \frac{-2t\left(\frac{t^2}{t^2-1}\right)}{(t^2-1)^2}$

$= \frac{-2t^3}{(t^2-1)^3}$

C) $x+y = 2t$

$x-y = \frac{2}{t}$

$t = \frac{2}{x-y}$

$x+y = 2t$

$x+y = 2\left(\frac{2}{x-y}\right)$

$x^2 - y^2 = 4$

$\frac{x^2}{4} - \frac{y^2}{4} = 1$

D) Hyperbolas.